Sequences

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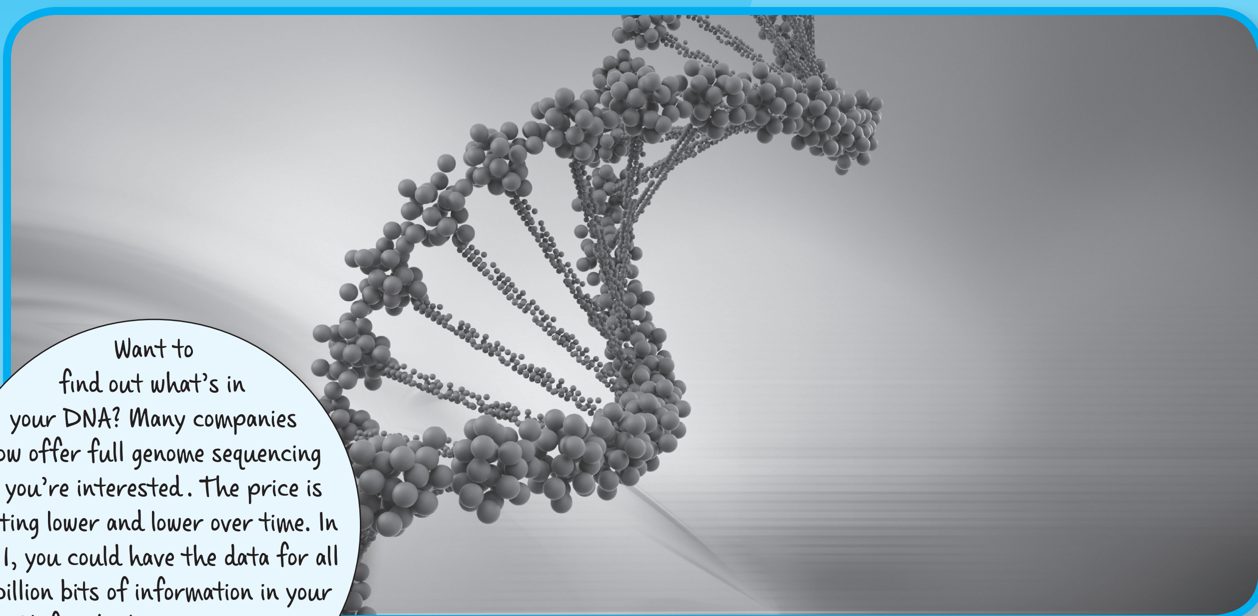



Want to find out what's in your DNA? Many companies now offer full genome sequencing if you're interested. The price is getting lower and lower over time. In 2011, you could have the data for all 6 billion bits of information in your DNA for the low price of $4000.

4.1 Is There a Pattern Here?

Recognizing Patterns and Sequences

LEARNING GOALS

In this lesson, you will:

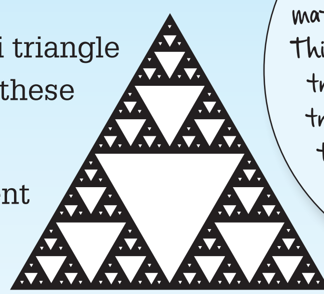
- Recognize patterns.
- Describe patterns.
- Represent patterns as sequences.
- Predict the next term in a sequence.

KEY TERMS

- sequence
- term of a sequence
- infinite sequence
- finite sequence

Want to play the chaos game? Dr. Michael Barnsley coined the phrase "chaos game," in which a pattern can be created by plotting a random point within a triangle and then rolling a number cube. A few hundred rolls of the number cube result in a pattern that forms what is called the Sierpinski triangle.

Care to see if you can create a Sierpinkski triangle by merely rolling a number cube? Follow these steps to see if you can do it.



1. First plot three points that will represent the vertices of a triangle. Label these points *A*, *B*, and *C*.

2. Plot a point anywhere inside the triangle.

3. Next, roll the number cube. If you roll a 1 or 2, measure half the distance from your initial point to vertex *A* and plot this point. If you roll a 3 or 4, measure half the distance from your initial point to vertex *B* and plot the point. If you roll a 5 or 6, measure half the distance from your initial point to vertex *C* and plot the point.

4. Repeat the process in Step 3, but this time, start with your new plotted point. Do this a few hundred times, and you may begin to create a Sierpinski triangle. Good luck!

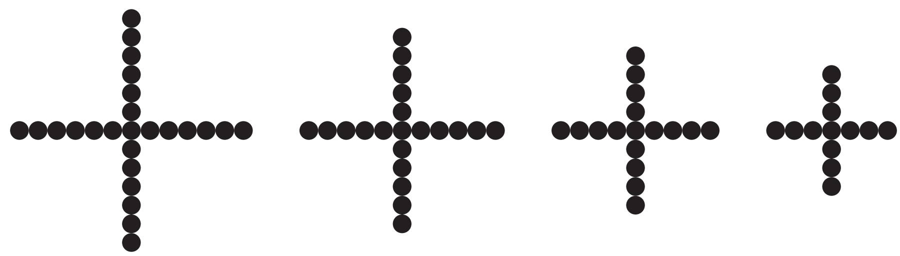
The Sierpinski triangle is named after the Polish mathematician Waclaw Sierpinski. This triangle consists of one large triangle, and within the larger triangle consists three smaller triangles, each of which also contain three smaller triangles, and so on.

PROBLEM 1 Do You See a Pattern?

A **sequence** is a pattern involving an ordered arrangement of numbers, geometric figures, letters, or other objects. A **term of a sequence** is an individual number, figure, or letter in the sequence.

Examples of sequences are shown. Describe the pattern, draw or describe the next terms, and represent each sequence numerically.

"Positive Thinking"



- Analyze the number of dots. Describe the pattern.

- Draw the next three figures of the pattern.

4

- Write the sequence numerically to represent the number of dots in each of the first 7 figures.

Family Tree

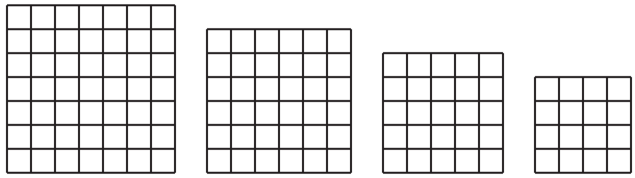
Jessica is investigating her family tree by researching each generation, or set, of parents. She learns all she can about the first four generations, which include her two parents, her parents' parents, her parents' parents' parents, and her parents' parents' parents' parents.

- Think about the number of parents. Describe the pattern.

- Determine the number of parents in the fifth and sixth generations.

- Write a numeric sequence to represent the number of parents in each of the 6 generations.

A Collection of Squares



- Analyze the number of small squares in each figure. Describe the pattern.

- Draw the next three figures of the pattern.

- Write the sequence numerically to represent the number of small squares in each of the first 7 figures.

4

Al's Omelets

Al's House of Eggs N'at makes omelets. Al begins each day with 150 eggs to make his famous *Bestern Western Omelets*. After making 1 omelet, he has 144 eggs left. After making 2 omelets, he has 138 eggs left. After making 3 omelets, he has 132 eggs left.

- Think about the number of eggs Al has left after making each omelet. Describe the pattern.

- Determine the number of eggs left after Al makes the next two omelets.

- Write the sequence numerically to represent the number of eggs left after Al makes each of the first 5 omelets. Include the number of eggs he started with.

Mario's Mosaic

Mario is creating a square mosaic in the school courtyard as part of his next art project. He begins the mosaic with a single square tile. Then he adds to the single square tile to create a second square made up of 4 tiles. The third square he adds is made up of 9 tiles, and the fourth square he adds is made up of 16 tiles.

- Think about the number of tiles in each square. Describe the pattern.

- Determine the number of tiles in the next two squares.

- Write the sequence numerically to represent the number of tiles in each of the first 6 squares.

Troop of Triangles

- Analyze the number of dark triangles. Describe the pattern.

- Draw the next two figures of the pattern.

- Write the sequence numerically to represent the number of dark triangles in each of the first 6 figures.

4

Gamer Guru

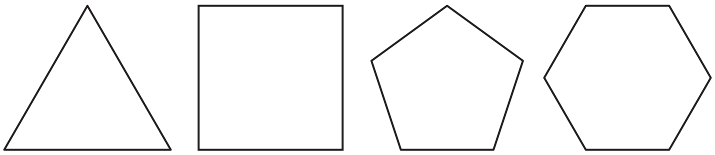
Mica is trying to beat his high score on his favorite video game. He unlocks some special mini-games where he earns points for each one he completes. Before he begins playing the mini-games (0 mini-games completed), Mica has 500 points. After completing 1 mini-game he has a total of 550 points, after completing 2 mini-games he has 600 points, and after completing 3 mini-games he has 650 points.

- Think about the total number of points Mica gains from mini-games. Describe the pattern.

- Determine Mica's total points after he plays the next two mini-games.

- Write the sequence numerically to represent Mica's total points after completing each of the first 5 mini-games. Include the number of points he started with.

Polygon Party



- Analyze the number of sides in each polygon. Describe the pattern.

- Draw the next two figures of the pattern.

- Write the sequence numerically to represent the number of sides of each of the first 6 polygons.

4

Pizza Contest

Jacob is participating in a pizza-making contest. Each contestant not only has to bake a delicious pizza, but they have to make the largest pizza they can. Jacob's pizza has a 6-foot diameter! After the contest, he plans to cut the pizza so that he can pass the slices out to share. He begins with 1 whole pizza. Then, he cuts it in half. After that, he cuts each of those slices in half. Then he cuts each of those slices in half, and so on.

- Think about the size of each slice in relation to the whole pizza. Describe the pattern.

- Determine the size of each slice compared to the original after the next two cuts.

- Write the sequence numerically to represent the size of each slice compared to the original after each of the first 5 cuts. Include the whole pizza before any cuts.

Coin Collecting

4

Miranda's uncle collects rare coins. He recently purchased an especially rare coin for $5. He claims that the value of the coin will triple each year. So even though the coin is currently worth $15, next year it will be worth $15. In 2 years it will be worth $45, and in 3 years it will be worth $135.

- Think about how the coin value changes each year. Describe the pattern.

- Determine the coin value after 4 years and after 5 years.

- Write the sequence numerically to represent the value of the coin after each of the first 5 years. Include the current value.

PROBLEM 2 What Do You Notice?

There are many different patterns that can generate a sequence of numbers. For example, you may have noticed that some of the sequences in Problem 1, *Do You See a Pattern?* were generated by performing the same operation using a constant number. In other sequences, you may have noticed a different pattern.

The next term in a sequence is calculated by determining the pattern of the sequence, and then using that pattern on the last known term of the sequence.

1. For each sequence in Problem 1, write the problem name and numeric sequence in the table shown. Also in the table, record whether the sequence increases or decreases, and describe the operation(s) used to create each sequence. The first one has been done for you.

Problem Name	Numeric Sequence	Increases or Decreases	Sequence Description
"Positive Thinking"	25, 21, 17, 13, 9, 5, 1	Decreases	Begin at 25. Subtract 4 from each term.

4

2. Which sequences are similar? Explain your reasoning.

PROBLEM 3 Do Sequences *Ever* End?

1. Consider a sequence in which the first term is 64, and each term after that is calculated by dividing the previous term by 4.

   Margaret says that this sequence ends at 1 because there are no whole numbers that come after 1. Jasmine disagrees and says that the sequence continues beyond 1. Who is correct? If Margaret is correct, explain why. If Jasmine is correct, predict the next two terms of the sequence.

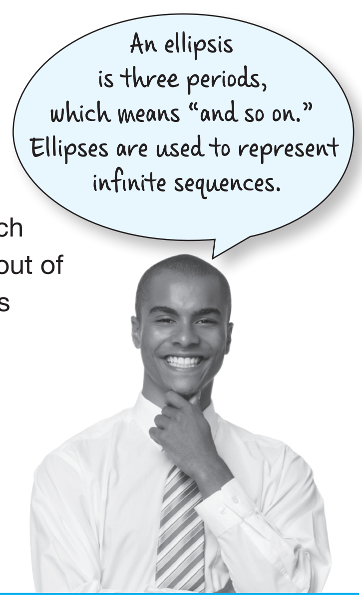
4

If a sequence continues on forever, it is called an **infinite sequence**. If a sequence terminates, it is called a **finite sequence**.

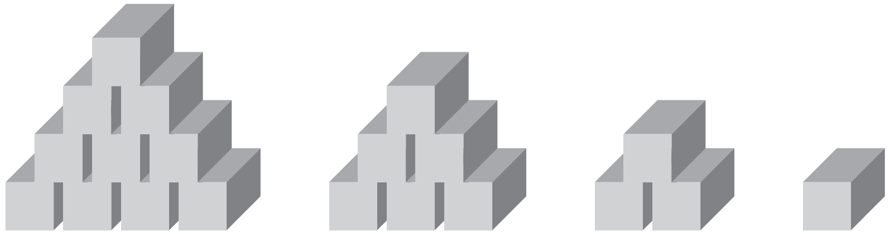
For example, consider an auditorium where the seats are arranged according to a specific pattern. There are 22 seats in the first row, 26 seats in the second row, 30 seats in the third row, and so on. Numerically, the sequence is 22, 26, 30, . . . , which continues infinitely. However, in the context of the problem, it does not make sense for the number of seats in each row to increase infinitely. Eventually, the auditorium would run out of space! Suppose that this auditorium can hold a total of 10 rows of seats. The correct sequence for this problem situation is:

22, 26, 30, 34, 38, 42, 46, 50, 54, 58.

Therefore, because of the problem situation, the sequence is a finite sequence.



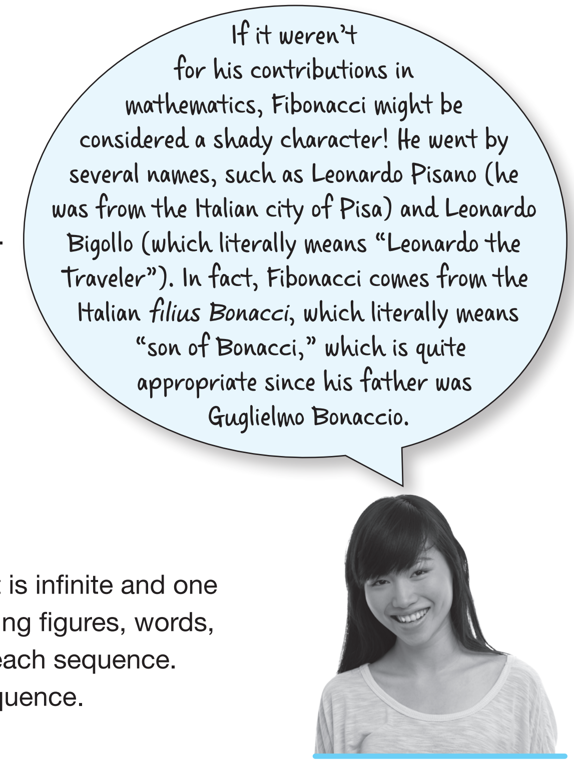
2. Does the pattern shown represent an infinite or finite sequence? Explain your reasoning.



3. One of the most famous infinite sequences is the Fibonacci sequence. The first 9 terms in the Fibonacci sequence are shown:

0, 1, 1, 2, 3, 5, 8, 13, 21, . . .

Explain in your own words the pattern that determines the Fibonacci sequence. Then, predict the next five terms in the sequence.



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4. Write your own two sequences—one that is infinite and one that is finite. Describe your sequence using figures, words, or numbers. Give the first four terms of each sequence. Explain how you know that each is a sequence.

Be prepared to share your solutions and methods.

4

4.2

The Password Is... Operations!

Arithmetic and Geometric Sequences

LEARNING GOALS

In this lesson, you will:

- Determine the next term in a sequence.
- Recognize arithmetic sequences.
- Determine the common difference.
- Recognize geometric sequences.
- Determine the common ratio.

KEY TERMS

- arithmetic sequence
- common difference
- geometric sequence
- common ratio

Nicknames and code names are often used for football plays, people—even our pets. But code names are no joke when it comes to important diplomats. In fact, the Secret Service assigns a code name to the president of the United States and the first family. Some classic code names for former U.S. presidents were "Tumbler" for President George Walker Bush, "Timberwolf" for President George Herbert Walker Bush, "Deacon" for President Jimmy Carter, and "Lancer" for President John Kennedy.

One of the most famous code names for a president was "Rawhide." In released Secret Service radio communications from the 1980s, the agents can be heard saying, "Rawhide is okay" after someone fired a gun at the president. However, it quickly became apparent that "Rawhide" was not okay and that he had in fact been shot during the assassination attempt. We now know that "Rawhide" was actually former President Ronald Reagan, who did survive the attempt on his life. Unfortunately, it was after that assassination attempt that codes names for presidents became public knowledge.

Presidents and other important diplomats still have code names, but since these have become public knowledge, they are more for tradition. You don't think that the Secret Service would let important things like code names become general knowledge, do you?

PROBLEM 1 What Comes Next, and How Do You Know?

1. Carefully cut out Sequences A through P. Make sure you do not cut away the letter representing the sequence.
2. Determine the unknown terms of each sequence. Describe the pattern under each sequence.
3. Sort the sequences into groups based on common characteristics. In the space provided, record the following information for each of your groups.
   - List the letters of the sequences in each group.
   - Provide a rationale as to why you created each group.

4

4. What mathematical operation(s) did you perform in order to determine the next terms of each sequence?

A

45, 90, 180, 360, ________ , ________ ,

________ , . . .

B

-4, -2, 0, 2, ________ , ________ ,

________ , . . .

C

-2, -6, -18, -54, ________ , ________ ,

________ , . . .

D

2, 5, 10, 17, ________ , ________ ,

________ , . . .

E

4, $\frac{7}{4}$, $-\frac{1}{2}$, $-\frac{11}{4}$, ________ , ________ ,

________ , . . .

F

1234, 123.4, 12.34, 1.234, ________ ,

________ , ________ , . . .

G

1, -2, 3, -4, 5, ________ , ________ ,

________ , ________ , . . .

H

-20, -16, -12, -8, -4, ________ ,

________ , ________ , . . .

4

4

I

$1, 10, 100, 1000,$ ________ , ________ , . . .

J

$-5, -\frac{5}{2}, -\frac{5}{4}, -\frac{5}{8},$ ________ , ________ , . . .

K

$6.5, 5, 3.5, 2,$ ________ , ________ ,

________ , . . .

L

$86, 85, 83, 80, 76,$ ________ , ________ , . . .

M

$-16, 4, -1, \frac{1}{4},$ ________ , ________ , . . .

N

$1473.2, 1452.7, 1432.2, 1411.7,$ ________ ,

________ , ________ , . . .

O

$\sqrt{5}, 2, \sqrt{3}, \sqrt{2}, 1, 0, \sqrt{-1}$ ________ ,

________ , . . .

P

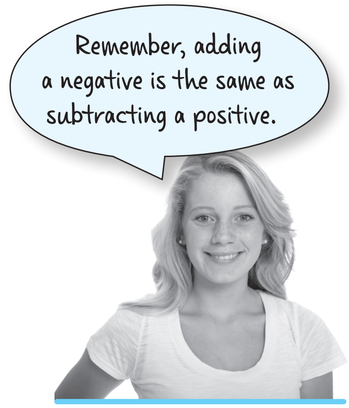
$-4, 12, -36, 108,$ ________ , ________ , . . .

4

4

PROBLEM 2 Arithmetic, My Dear Watson!

You can describe a pattern as adding a constant to, or subtracting a constant from each term to determine the next term for some sequences. For other sequences, you can describe the pattern as multiplying or dividing each term by a constant to determine the next term. Still other sequences cannot be described either way.



An **arithmetic sequence** is a sequence of numbers in which the difference between any two consecutive terms is a constant. In other words, it is a sequence of numbers in which a positive or negative constant is added to each term to produce the next term. This positive or negative constant is called the **common difference**. The common difference is typically represented by the variable *d*.

The common difference of a sequence is positive if the same *positive number* is added to each term to produce the next term. The common difference of a sequence is negative if the same *negative number* is added to each term to produce the next term.

4

Consider the sequence shown.

11, 9, 7, 5, …

The pattern is to add the same negative number, −2, to each term to determine the next term.

add −2 add −2 add −2

Sequence: 11, 9, 7, 5, . . .

This sequence is arithmetic and the common difference *d* is −2.

1. Suppose a sequence has the same starting number as the sequence in the worked example, but its common difference is 4.

   a. How would the pattern change?

   b. Is the sequence still arithmetic? Why or why not?

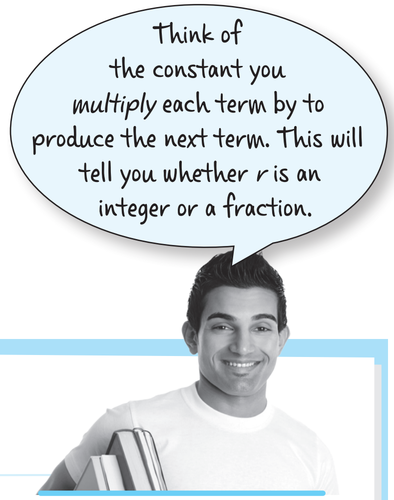
   c. If possible, write the first 5 terms of the new sequence.

2. Analyze the sequences you cut out in Problem 1, *What Comes Next, and How Do You Know?*

   a. List those sequences that are arithmetic.

   b. Write the common difference on each arithmetic sequence you cut out.

A **geometric sequence** is a sequence of numbers in which the ratio between any two consecutive terms is a constant. In other words, it is a sequence of numbers in which you multiply each term by a constant to determine the next term. This integer or fraction constant is called the **common ratio**. The common ratio is represented by the variable r.



Consider the sequence shown.

1, 2, 4, 8, . . .

The pattern is to multiply each term by the same number, 2, to determine the next term.

multiply by 2 — multiply by 2 — multiply by 2

Sequence: 1, 2, 4, 8, . . .

This sequence is geometric and the common ratio r is 2.

4

3. Suppose a sequence has the same starting number as the sequence in the worked example, but its common ratio is 3.

   a. How would the pattern change?

   b. Is the sequence still geometric? Explain your reasoning.

   c. If possible, write the first 5 terms for the new sequence.

4. Suppose a sequence has the same starting number as the sequence in the worked example, but its common ratio is $\frac{1}{3}$.

   a. How would the pattern change?

   b. Is the sequence still geometric? Why or why not?

   c. If possible, write the first 6 terms for the new sequence.

5. Suppose a sequence has the same starting number as the sequence in the worked example, but its common ratio is -2.

   a. How would the pattern change?

   b. Is the sequence still geometric? Explain your reasoning.

   c. If possible, write the first 6 terms for the new sequence.

4

6. Consider the sequence shown.

   270, 90, 30, 10, . . .

   Devon says that he can determine each term of this sequence by multiplying each term by $\frac{1}{3}$, so the common ratio is $\frac{1}{3}$. Chase says that he can determine each term of this sequence by dividing each term by 3, so the common ratio is 3. Who is correct? Explain your reasoning.

7. Analyze the sequences you cut out in Problem 1, *What Comes Next, and How Do You Know*? again.

   a. List those sequences that are geometric.

   b. Write the common ratio on each geometric sequence you cut out.

8. Consider the sequences from Problem 1 that are neither arithmetic nor geometric.

   a. List these sequences.

   b. Explain why these sequences are neither arithmetic nor geometric.

4

9. Consider the first two terms of a sequence.

3, 6, . . .

Dante says, "This is how I wrote the sequence for the given terms."

3, 6, 9, 12, . . .

Kira says, "This is the sequence I wrote."

3, 6, 12, 24, . . .

Who is correct? Explain your reasoning.

10. Using the terms given in Question 10, write a sequence that is neither arithmetic nor geometric. Then, have your partner tell you what the pattern is in your sequence.

4

11. How many terms did your partner need before the pattern was recognized?

12. Consider the sequence 2, 2, 2, 2, 2. . . Identify the type of sequence it is and describe the pattern.

Be prepared to share your solutions and methods.

The Power of Algebra Is a Curious Thing

4.3

Using Formulas to Determine Terms of a Sequence

LEARNING GOALS

In this lesson, you will:

- Write an explicit formula for arithmetic and geometric formulas.
- Write a recursive formula for arithmetic and geometric formulas.
- Use formulas to determine unknown terms of a sequence.

KEY TERMS

- index
- explicit formula
- recursive formula

Humans and tools go together like hand and glove. Some scientists claim that tools helped humans dominate other animals in ancient times. This makes sense—how else could humans have caught animals for food?

What about those ancient vegetarians? Yup, they used tools too—to till the soil and make it fertile for growing crops. Humans use tools for mathematics as well. Algebraic thinking has been around for centuries—in fact, it has been around for such a long period of time that two different people are associated with being the "father of algebra." No matter, algebra led to other tools like the abacus, the modern graphing calculator, and even the computer.

In much the same way, formulas were some of the first "tools" used to help humans calculate more quickly. And you probably guessed it: formulas can help you determine any unknown term in a sequence!

PROBLEM 1 Can I Get a Formula?

While a common ratio or a common difference can help you determine the next term in a sequence, how can they help you determine the thousandth term of a sequence? The ten-thousandth term of a sequence? Consider the sequence represented in the given problem scenario.

1. Rico owns a sporting goods store. He has agreed to donate $125 to the Centipede Valley High School baseball team for their equipment fund. In addition, he will donate $18 for every home run the Centipedes hit during the season. The sequence shown represents the possible dollar amounts that Rico could donate for the season.

125, 143, 161, 179, . . .

a. Identify the sequence type. Describe how you know.

b. Determine the common ratio or common difference for the given sequence.

c. Complete the table of values. Use the number of home runs the Centipedes could hit to identify the term number, and the total dollar amount Rico could donate to the baseball team.

Number of Home Runs	Term Number (n)	Donation Amount (dollars)
0	1	
1		
2		
3		
4		
5		
6		
7		
8		
9		

4

d. Explain how you can calculate the tenth term based on the ninth term.

e. Determine the 20th term. Explain your calculation.

f. Is there a way to calculate the 20th term without first calculating the 19th term? If so, describe the strategy.

g. Describe a strategy to calculate the 93rd term.

You can determine the 93rd term of the sequence in Question 1 by calculating each term before it, and then adding 18 to the 92nd term, but this will probably take a while! A more efficient way to calculate any term of a sequence is to use a formula.

Analyze the table. The examples shown are from the sequence showing Rico's contribution to the Centipedes baseball team in terms of home runs hit.

General Rule	Example
A lowercase letter is used to name a sequence.	a
The first term, or initial term, is referred to as a_1.	$a_1 = 125$
The remaining terms are named according to the term number.	$a_2 = 143$, $a_3 = 161, \ldots$
A general term of the sequence is referred to as a_n, also known as the *n*th term, where n represents the *index*.	a_n
The term previous to a_n is referred to as a_{n-1}.	a_{n-1}
The common difference is represented as d.	$d = 18$

The **index** is the position of the term (its term number) in a sequence.

2. What is a_3 in the sequence representing Rico's possible donation amount?

From these rules, you can develop a formula so that you do not need to determine the value of the previous term to determine subsequent terms.

An **explicit formula** of a sequence is a formula for calculating the value of each term of a sequence using the term's position in the sequence. The explicit formula for determining the *n*th term of an arithmetic sequence is:

$$\overset{n\text{th term}}{a_n} = \underset{\text{1st term}}{a_1} + \overset{\text{common difference}}{d}\underbrace{(n-1)}_{\text{previous term number}}$$

Consider the explicit formula to determine the 93rd term in this problem situation.

$$a_n = a_1 + d(n - 1)$$
$$a_{93} = 125 + 18(93 - 1)$$

where a_{93} represents the 93rd term, a_1 represents the first term (which is 125), the common difference d is 18, and the previous term from 93 is $(93 - 1)$.

$$a_{93} = 125 + 18(92)$$
$$a_{93} = 125 + 1656$$
$$a_{93} = 1781$$

The 93rd term of the sequence is 1781.

This means Rico will contribute a total of \$1781 if the Centipedes hit 92 home runs.

Remember that the 1st term in this sequence is the amount Rico donates if the team hits 0 home runs. So, the 93rd term represents the amount Rico donates if the team hits 92 home runs.

3. Use the explicit formula to determine the amount of money Rico will contribute if the Centipedes hit:

a. 35 home runs.

b. 48 home runs.

c. 86 home runs.

d. 214 home runs.

Remember, the term number is not the same as the number of home runs!

4. Rico decides to increase his initial contribution and amount donated per home run hit. He decides to contribute \$500 and will donate \$75.00 for every home run the Centipedes hit. Determine Rico's contribution if the Centipedes hit:

a. 11 home runs.

b. 26 home runs.

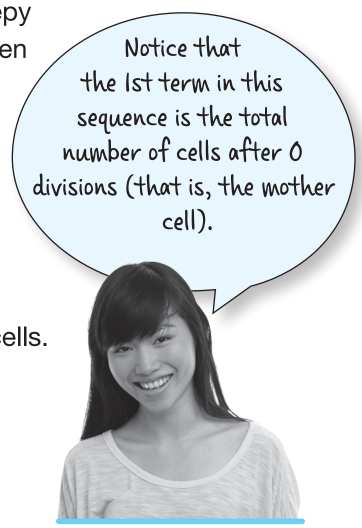
c. 39 home runs.

d. 50 home runs.

e. Write the first 10 terms of the sequence representing the new contribution Rico will donate to the Centipedes.

PROBLEM 2 They're Just Out of Control—But That's A Good Thing!

When it comes to bugs, bats, spiders, and—ugh, any other creepy crawlers—finding one in your house is finding *one* too many! Then again, when it comes to cells, the more the better! Animals, plants, fungi, slime, molds, and other living creatures consist of eukaryotic cells. During growth, generally there is a cell called a "mother cell" that divides itself into two "daughter cells." Each of those daughter cells then divides into two more daughter cells, and so on.



1. The sequence shown represents the growth of eukaryotic cells.

$$1, 2, 4, 8, 16, \ldots$$

a. Describe why this sequence is geometric.

b. Determine the common ratio for the given sequence.

c. Complete the table of values. Use the number of cell divisions to identify the term number, and the total number of cells after each division.

Number of Cell Divisions	Term Number (n)	Total Number of Cells
0	1	
1		
2		
3		
4		
5		
6		
7		
8		
9		

d. Explain how you can calculate the tenth term based on the ninth term.

e. Determine the 20th term. Explain your calculation.

f. Is there a way to calculate the 20th term without first calculating the 19th term? If so, describe the strategy.

As you discovered in Problem 1, *Can I Get a Formula?* a more efficient way to calculate any term of an arithmetic sequence is to use an explicit formula. You can also use an explicit formula for geometric sequences.

Analyze the table shown. The examples are from the sequence showing eukaryotic cell growth.

General Rule	Example
A lowercase letter is used to name a sequence.	g
The first term, or initial term, is referred to as g_1.	$g_1 = 1$
The remaining terms are named according to the term number.	$g_2 = 2$, $g_3 = 4, \ldots$
A general term of the sequence is referred to as g_n, also known as the *n*th term.	g_n
The term previous to g_n is referred to as g_{n-1}.	g_{n-1}
The common ratio is represented as r.	$r = 2$

2. What is g_3 in the sequence representing eukaryotic cell growth?

The explicit formula for determining the *n*th term of a geometric sequence is:

$$\overset{n\text{th term}}{g_n} = \underset{\text{1st term}}{g_1} \cdot \underset{\text{common ratio}}{r}^{\overbrace{n-1}^{\text{previous term number}}}$$

Consider the explicit formula to determine the 20th term in this problem situation.

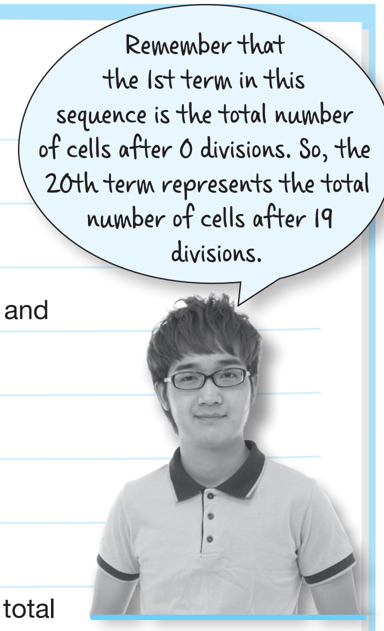
$$g_n = g_1 \cdot r^{n-1}$$
$$g_{20} = 1 \cdot 2^{20-1}$$

where g_{20} represents the 20th term, g_1 represents the first term (which is 1), the common ratio r is 2, and $^{20-1}$ represents the previous term number.

$$g_{20} = 1 \cdot 2^{19}$$
$$g_{20} = 1 \cdot 524{,}288$$
$$g_{20} = 524{,}288$$

The 20th term of the sequence is 524,288.

This means that after 19 cell divisions, there are a total of 524,288 cells.



3. Use the explicit formula to determine the total number of cells after:

a. 11 divisions.

b. 14 divisions.

c. 18 divisions.

d. 22 divisions.

4. Suppose that a scientist has 5 eukaryotic cells in a petri dish. She wonders how the growth pattern would change if each mother cell divided into 3 daughter cells. For this situation, determine the total number of cells in the petri dish after:

a. 4 divisions.

b. 7 divisions.

c. 13 divisions.

d. 16 divisions.

4

e. Write the first 10 terms of the sequence for the scientist's hypothesis.

PROBLEM 3 So, You've Explicitly Determined Terms, But Is There Another Way?

The explicit formula is very handy for determining terms of a sequence, but is there another way?

A **recursive formula** expresses each new term of a sequence based on the preceding term in the sequence. The recursive formula for determining the *n*th term of an arithmetic sequence is:

$$\overset{n\text{th term}}{a_n} = \underbrace{a_{n-1}}_{\text{previous term}} + \overset{\text{common difference}}{d}$$

Notice that you do not need to know the first term when using the recursive formula. However, you need to know the previous term to determine the next term. This is why this formula is commonly referred to as the NOW NEXT formula.

Consider the sequence shown.

$$-2, -9, -16, -23, \ldots$$

Use the recursive formula to determine the 5th term.

$$a_n = a_{n-1} + d$$

$$a_5 = a_{5-1} + (-7)$$

where a_5 represents the 5th term, a_{5-1} represents the previous term (which is -23), and the common difference d is -7.

$$a_5 = a_4 + (-7)$$

$$a_5 = -23 + (-7)$$

$$a_5 = -30$$

The 5th term of the sequence is -30.

4

The recursive formula for determining the *n*th term of a geometric sequence is:

*n*th term — common ratio

$$g_n = \underbrace{g_{n-1}}_{\text{previous term}} \cdot r$$

Consider the sequence shown.

4, 12, 36, 108, . . .

Use the recursive formula to determine the 5th term.

$$g_n = g_{n-1} \cdot r$$

$$g_5 = g_{5-1} \cdot (3)$$

where g_5 represents the 5th term, g_{5-1} represents the previous term (which is 108), and the common ratio r is 3.

$$g_5 = g_4 \cdot (3)$$

$$g_5 = 108 \cdot (3)$$

$$g_5 = 324$$

The 5th term of the sequence is 324.

1. Determine whether each sequence is arithmetic or geometric. Then use the recursive formula to determine the unknown term in each sequence.

a. $\frac{5}{3}$, 5, 15, 45, ________ , . . .

b. −45, −61, −77, −93, ________ , . . .

c. $-3, 1,$ ________ $9, 13, \ldots$

d. $-111, 222,$ ________, $888, -1776, \ldots$

e. $-30, -15,$ ________, $-3.75, -1.875,$ ________, . . .

f. 3278, 2678, 2078, ________, ________, ________, . . .

4

2. Consider the sequence in Question 1 part (f).

a. Use the recursive formula to determine the 9th term.

b. Use the explicit formula to determine the 9th term.

c. Which formula do you prefer? Why?

d. Which formula would you use if you wanted to determine the 61st term of the sequence? Explain your reasoning.

Let's explore how to use a recursive formula on a graphing calculator to determine the 20th term in sequence *t*.

Consider the sequence *t*: 3, 10, 17, 24, 31, . . .

You can use a graphing calculator to generate terms in a sequence using a recursive formula.

Step 1: Enter the first value of the sequence. Then press **ENTER** to register the first term. The calculator can now recall that first term.

Step 2: From that term, add the common difference. Press **ENTER**. The next term should be calculated. The calculator can now recall the formula as well.

Step 3: Press **ENTER** and the next term should be calculated.

Step 4: Continue pressing **ENTER** until you determine the *n*th term of the sequence you want to determine.

Press 3 then ENTER since 3 is the first term.

The common difference is 7.

Keep track of how many times you press ENTER so you know when you have the 20th term!

4

3. Determine the 20th term of sequence *t*. How did you determine this term?

The given calculator instructions can help you identify the term of a sequence. However, you might have found it challenging to keep track of the term numbers when calculating the 20th term. Another way to determine the 20th term in sequence *t* is to use a graphing calculator to generate 2 sequences at the same time. The first sequence will keep track of the term number, and the second sequence will generate the term value.

You can use a graphing calculator to generate two sequences at the same time in order to determine a certain term in a sequence.

For sequence *t*, the first term number is 1, and the first value is 3. Press 2nd {1,3 2nd} ENTER. The calculator will display {1 3}.

Step 1: Within a set of brackets, enter the first term number followed by a comma and then the first term value of the sequence. The **2ND** key is used to enter the brackets. Press **ENTER**.

Step 2: Provide direction to the calculator on how to generate each term of the sequence.

Press **2ND{2NDANS(1)** and then indicate how the term numbers will increase or decrease, and by how much by entering the plus or minus sign and the amount of increase or decrease.

Then press **,2ND ANS(2)** and enter the common difference of the term values. Then close the brackets by pressing **2ND}** and press **ENTER**.

In sequence *t*, each term value increases by 1 and the common difference is 7. Enter this information by pressing 2nd{2nd ANS(1)+1, 2nd ANS(2) + 7 2nd} ENTER. The calculator will display {2 10}

The calculator will display the next term number and value.

Ok . . . so after I pressed **ENTER**, {3 17} was displayed. Is that what you got?

Step 3: Press **ENTER** and the next term number and value will be displayed.

Step 4: Continue pressing **ENTER** until you reach the *n*th term number and value you want to determine.

The values are much easier to keep track of when the term number is displayed as well as the value!

4. Does your solution using this method match your solution in Question 3?

5. Use a graphing calculator to determine each solution.

   a. Identify the seventh term of this arithmetic sequence: 6, 14, 22, . . .

   b. List the first 10 terms of this arithmetic sequence: 54, 47, 40, . . .

   c. List the first 10 terms of the arithmetic sequence generated by this recursive formula:

   $t_1 = 8$

   $t_n = t_{n-1} + 19$

   d. Identify the 30th term of this arithmetic sequence: 45, 51, 57, . . .?

Talk the Talk

1. Explain the advantages and disadvantages of using the explicit formula.

4

2. Explain the advantages and disadvantages of using the recursive formula.

Be prepared to share your solutions and methods.

4.4

Thank Goodness Descartes Didn't Drink Some Warm Milk!

Graphs of Sequences

LEARNING GOALS

In this lesson, you will:

- Graph arithmetic sequences.
- Graph geometric sequences.
- Recognize graphical behavior of sequences.
- Sort sequences that are represented graphically.

You've worked with coordinate planes before, but you may not know how they were invented. As one story goes, the 16th century French mathematician and philosopher René Descartes (pronounced day-Kart) was suffering through a bout of insomnia. While attempting to fall asleep, he spotted a fly walking on the tiled ceiling above his head. At this sight, his mind began to wander and a question popped in his head: Could he describe the fly's path without tracing the *actual* path?

From that question came the revolutionary invention of the coordinate system—an invention which makes it possible to link algebra and geometry. Where have you seen examples of coordinate planes? How do coordinate planes help you identify the locations of objects?

PROBLEM 1 Sequences as Tables and Graphs

Sometimes writers can have words flow from mind to paper without any struggles. However, that is not usually the case. For the most part, writers need to organize their thoughts, and many times they use outlines to organize these thoughts. Some of the same struggles may arise in mathematics, especially when dealing with sequences. Thus, creating a table to organize values can help you determine the sequence.

1. Consider sequence *a* represented by the explicit formula shown.

$$a_n = -10 + 4(n - 1)$$

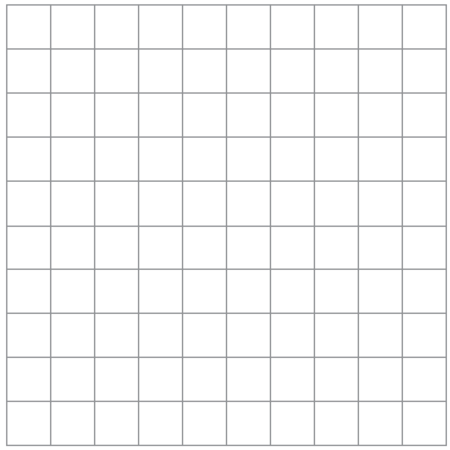
a. Complete the table for sequence *a*.

Term Number (*n*)	Term Value
1	−10
2	
3	
4	
5	
6	
7	
8	
9	
10	

b. Write each pair of numbers from the table as an ordered pair. Let the independent variable represent the term number, and let the dependent variable represent the term value.

4

c. Graph the ordered pairs on the grid shown and label the axes.



d. Describe the shape of the graph.

e. Is the graph discrete or continuous? Explain your reasoning.

f. Can you use the graph to predict the 20th term? Explain your reasoning.

2. Consider sequence g represented by the explicit formula shown.

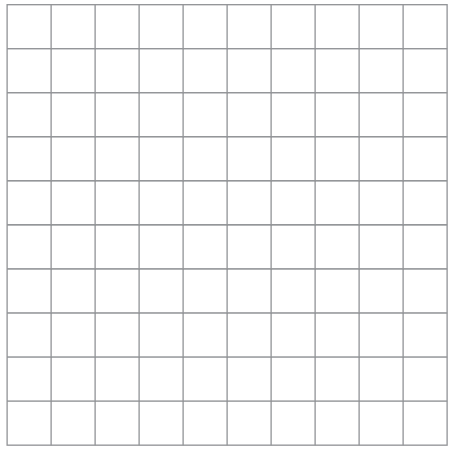
$$g_1 = 1$$
$$g_n = 2^{n-1}$$

a. Create a table of values using the first ten terms of sequence g.

Term Number (n)	Term Value

b. Write each pair of numbers from the table as an ordered pair. Let the independent variable represent the term number, and let the dependent variable represent the term value.

c. Graph the ordered pairs on the grid shown and label the axes.



d. Describe the shape of the graph.

4

e. Is the graph discrete or continuous? Explain your reasoning.

f. Can you use the graph to predict the 20th term? Explain your reasoning.

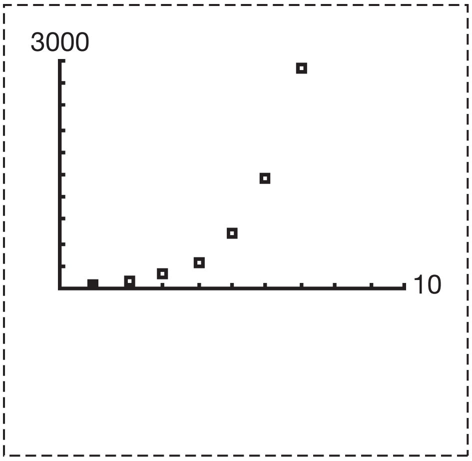
PROBLEM 2 Match Up the Graphs!

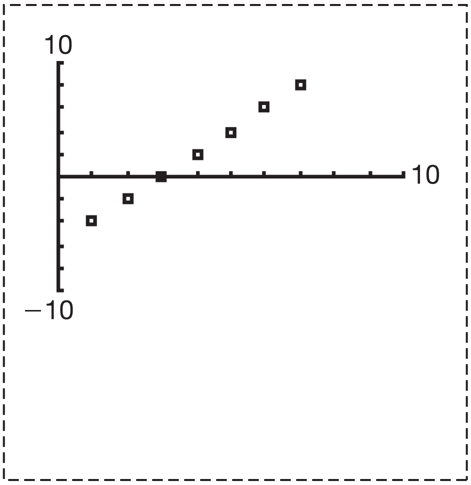
As you have already discovered when studying functions, graphs can help you see trends of a sequence—and at times can help you predict the next term in a sequence.

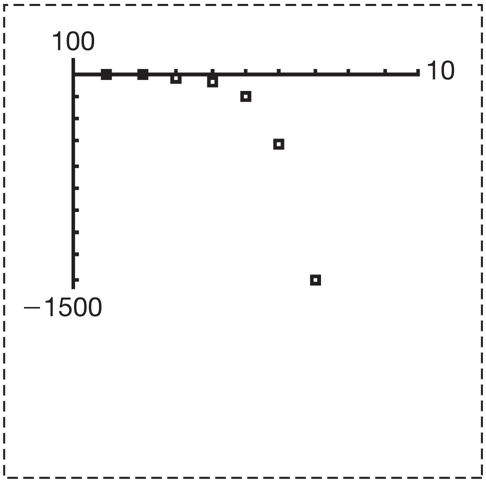
1. Create graphic organizers to identify different arithmetic and geometric sequences.

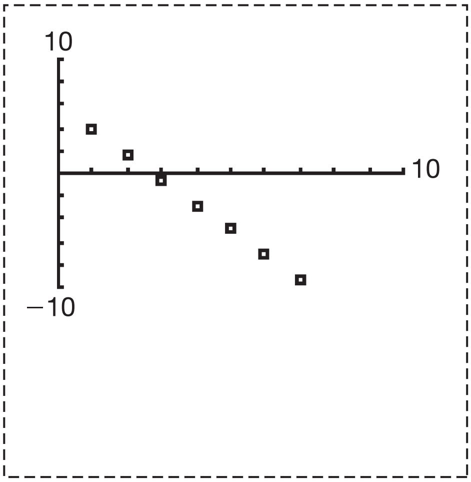
   - Carefully remove the 12 graphic organizers at the end of this lesson.
   - Gather the arithmetic and geometric sequences from Lesson 4.2, *The Password Is . . . Operations!* Paste one sequence in the "Sequence" section of each graphic organizer.
   - Write the explicit formula for the sequence in the "Explicit Formula" section of each graphic organizer.
   - Write the recursive formula for the sequence in the "Recursive Formula" section of each graphic organizer.
   - The graphs representing the arithmetic and geometric sequences from Lesson 4.2 are located on the following pages. Cut out these graphs. Match each graph to its appropriate sequence and paste it into the "Graph" section of each graphic organizer.
   - In the center of each graphic organizer, write the sequence type (arithmetic or geometric).

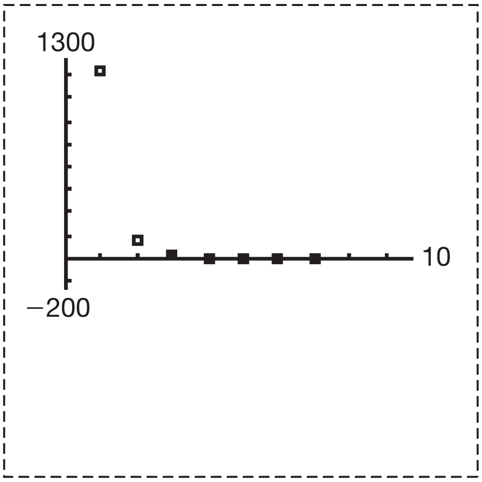
4

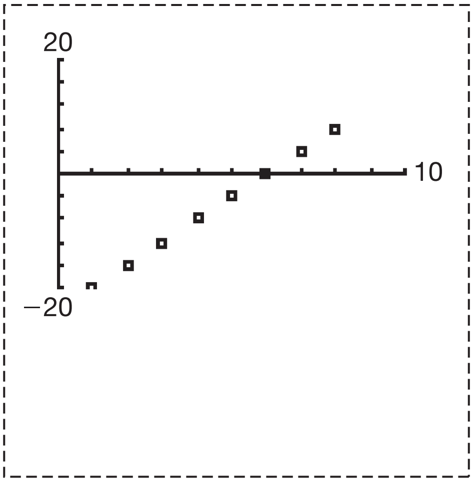












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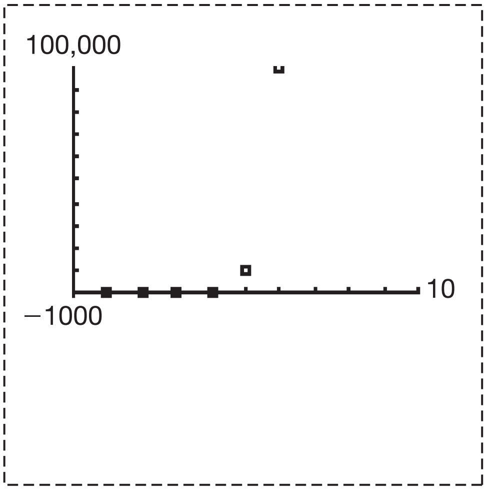

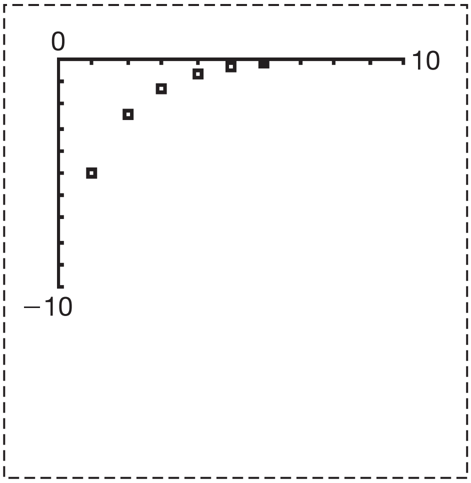

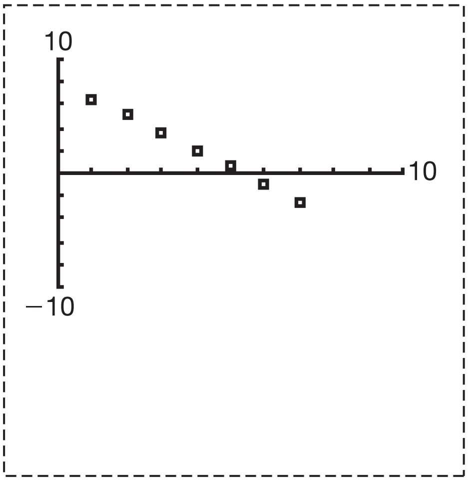

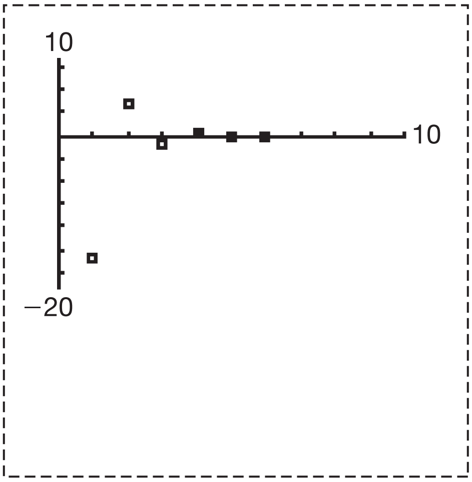

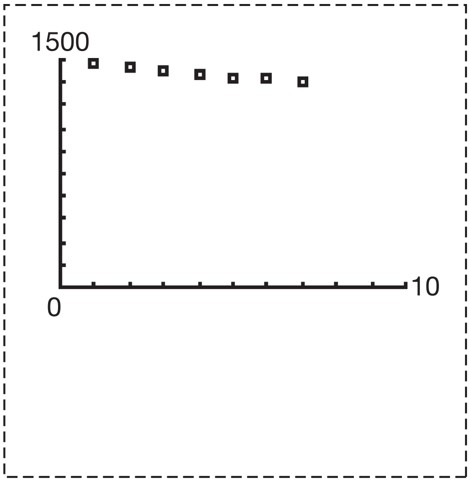

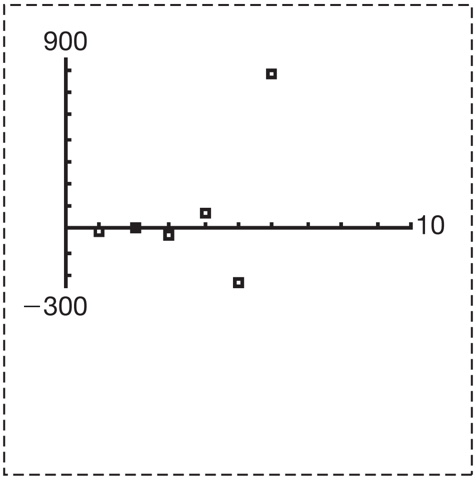

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2. Did knowing whether the sequences were increasing or decreasing help you to match the graphs to their corresponding sequences? Explain your reasoning.

3. What other strategies did you use to match the graphs to their corresponding sequences?

Be prepared to share your solutions and methods.

4

Sequence

Explicit Formula

Sequence Type:

Graph

Recursive Formula

Sequence

Explicit Formula

Sequence Type:

Graph

Recursive Formula

Sequence

Explicit Formula

Sequence Type:

Graph

Recursive Formula

4

Sequence

Explicit Formula

4

Sequence Type:

Graph

Recursive Formula

Sequence

Explicit Formula

Sequence Type:

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Recursive Formula

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Sequence

Explicit Formula

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Recursive Formula

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Sequence

Explicit Formula

Sequence Type:

Graph

Recursive Formula

Sequence

Explicit Formula

Sequence Type:

Graph

Recursive Formula

4

Sequence

Explicit Formula

Sequence Type:

Graph

Recursive Formula

4.5

Well, Maybe It *Is* a Function!

Sequences and Functions

LEARNING GOALS

In this lesson, you will:

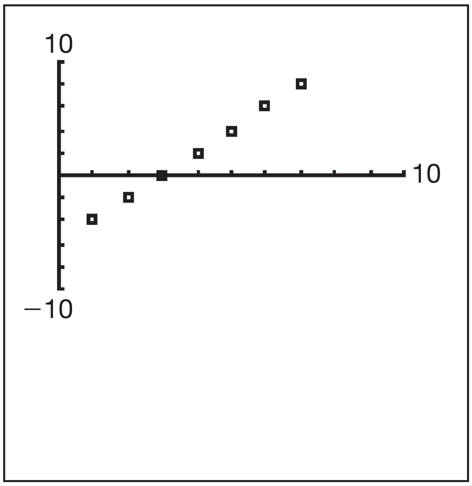
- Write an arithmetic sequence as a linear function.
- Make the connection between the graph of an arithmetic sequence, and the graph of a linear function.
- Write a geometric sequence as an exponential function.
- Make the connection between the graph of a geometric sequence, and the graph of an exponential function.
- Contrast an exponential function and a geometric sequence with a negative common ratio.

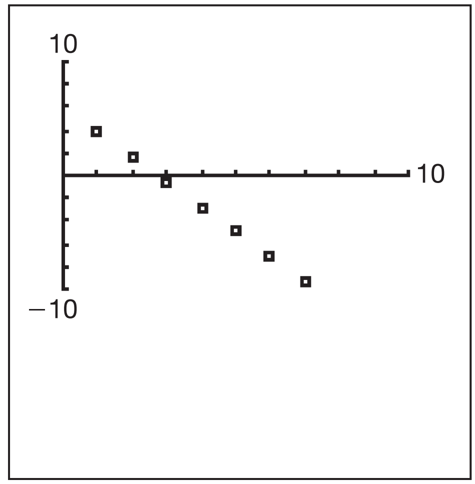
You might have heard the saying "If it looks like a duck and walks like a duck, it's probably a duck." The meaning is simple: if an object has some characteristics of something familiar, well, then it must be that familiar object, right? That seems pretty simple, but actually, it can be rather difficult. Attorneys and the law may counter with: "you can't judge a book by its cover."

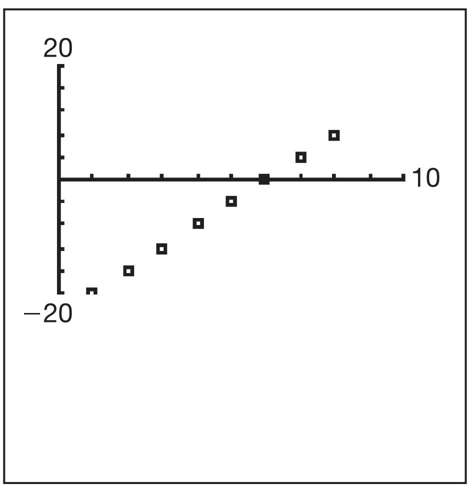
You have just encountered conjecture and proof through these two sayings. In mathematics, just like in law, more is needed than just a conjecture (a statement). A proof is needed. You will learn that in geometry, conjectures and proofs are a very important. But guess what? You're about to get an early dose of conjecture and proof!

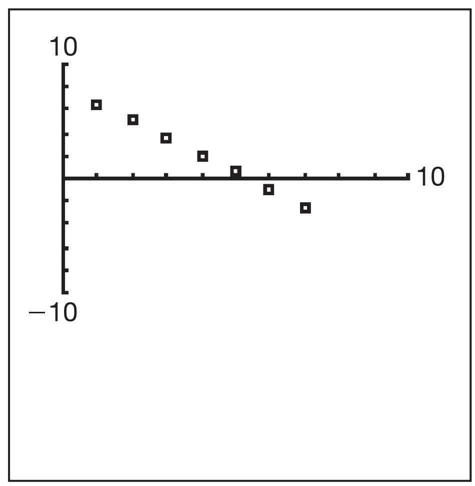
PROBLEM 1 If It Looks Like a Function and Quacks Like a Function…

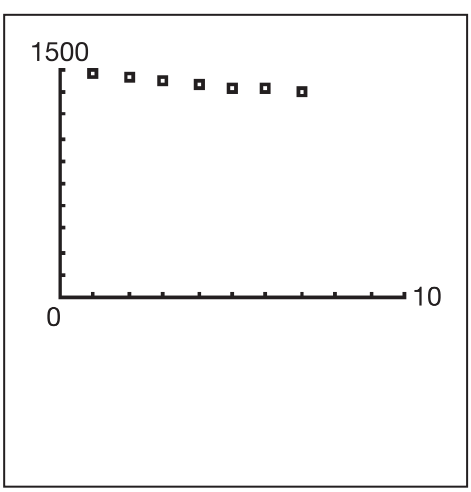
The graphs of the arithmetic sequences from Lesson 4.4, *Thank Goodness Descartes Didn't Drink Some Warm Milk!* are shown.











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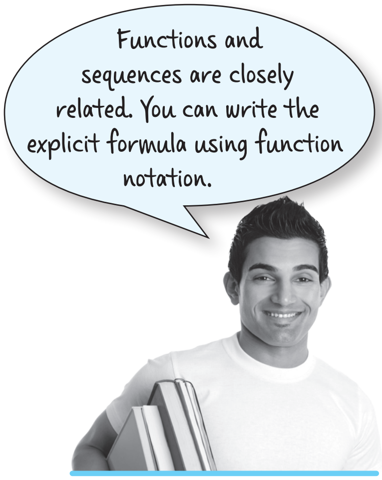
1. Identify the function family that represents the graphs of the arithmetic sequences shown. Do you think all arithmetic sequences belong to this function family? Explain your reasoning.



Consider the explicit formula for the arithmetic sequence shown in the first graph.

$$a_n = -4 + 2(n - 1)$$

You *can* write the explicit formula for the arithmetic sequence

$$a_n = -4 + 2(n - 1)$$

in function notation.

Statement	**Reason**
$a_n = -4 + 2(n - 1)$	Explicit Formula for Arithmetic Sequence
$f(n) = -4 + 2(n - 1)$	Represent a_n using function notation.
$f(n) = -4 + 2n - 2$	Distributive Property
$f(n) = 2n - 2 - 4$	Commutative Property
$f(n) = 2n - 6$	Associative Property

So $a_n = -4 + 2(n - 1)$ written in function notation is $f(n) = 2n - 6$.

2. In the Lesson 4.4, *Thank Goodness Decartes Didn't Drink Warm Milk!* you created graphic organizers that identified the explicit formulas for four arithmetic sequences. Rewrite each explicit formula in function notation.

a. Sequence E

$a_n = 4 + \left(-\frac{9}{4}\right)(n - 1)$

b. Sequence H

$a_n = -20 + 4(n - 1)$

c. Sequence K

$a_n = 6.5 + (-1.5)(n - 1)$

d. Sequence N

$a_n = 1473.2 + (-20.5)(n - 1)$

4

3. Based on the formulas, identify the function family of these arithmetic sequences. Explain your reasoning.

4. What is the relationship between the common difference of an arithmetic sequence and the slope of a linear function?

5. Hank says that the y-intercept of a linear function is the same as the first term of an arithmetic sequence. Is he correct? Why or why not?

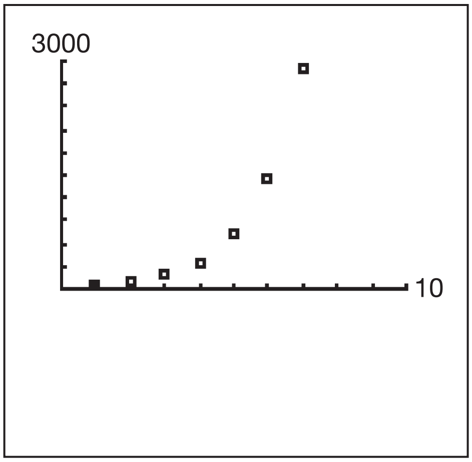
6. Represent the y-intercept of an arithmetic sequence algebraically.

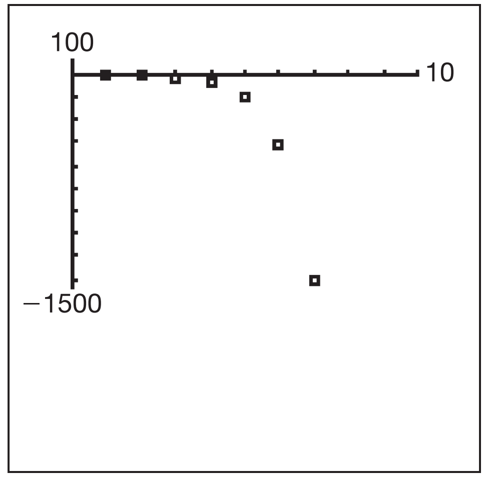
7. Complete the table by writing each part of the linear function that corresponds to each part of the arithmetic sequence.

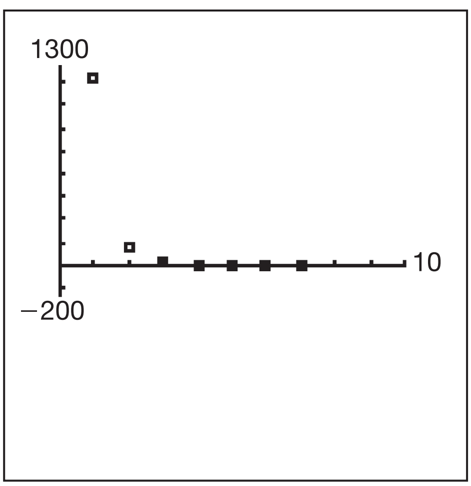
Arithmetic Sequence	**Linear Function**
$a_n = a_1 + d(n - 1)$	$f(x) = mx + b$
a_n	
d	
n	
$a_1 - d$	

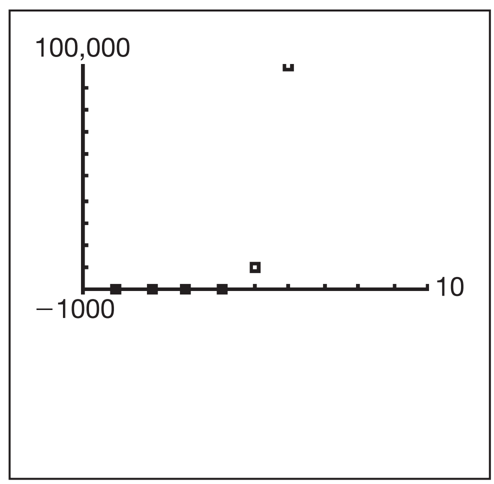
PROBLEM 2 And If It Swims Like a Function and Smells Like a Function…

The graphs of the geometric sequences from Lesson 4.4, *Thank Goodness Descartes Didn't Drink Some Warm Milk!* are shown.









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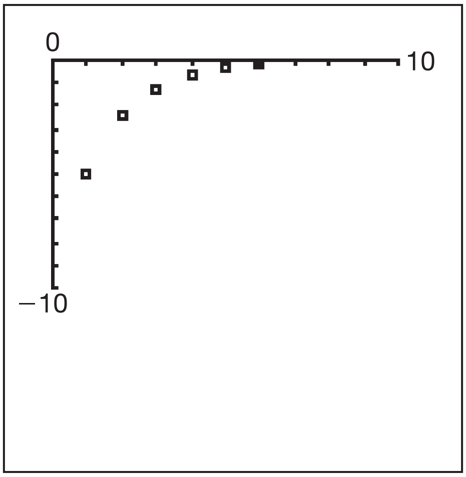

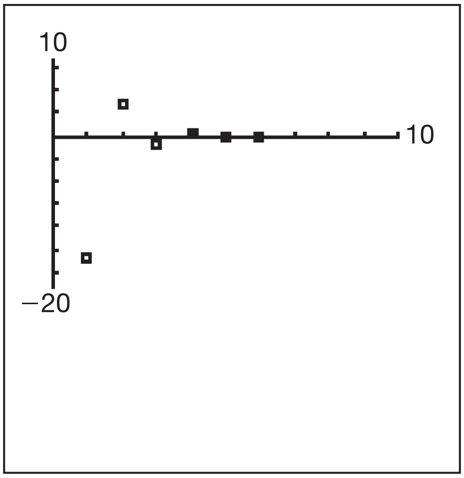

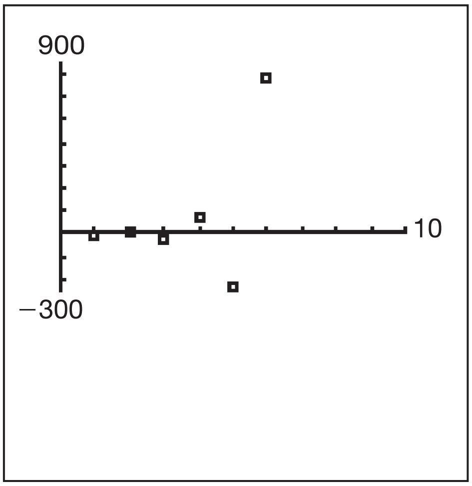

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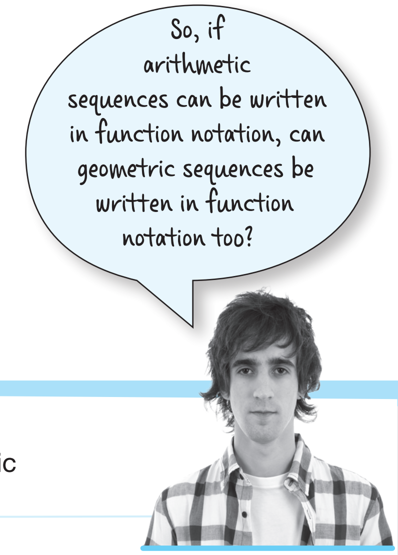
1. Do all of the graphs of the geometric sequences belong to the same function family? Why or why not?

Consider the explicit formula for the geometric sequence shown in the first graph.

$$g_n = 45 \cdot 2^{n-1}$$



You *can* write the explicit formula for the geometric sequence $g_n = 45 \cdot 2^{n-1}$ in function notation.

Statement	Reason
$g_n = 45 \cdot 2^{n-1}$	Explicit Formula for Geometric Sequence
$f(n) = 45 \cdot 2^n \cdot 2^{-1}$	Product Rule of Powers
$f(n) = 45 \cdot 2^{-1} \cdot 2^n$	Commutative Property
$f(n) = 45 \cdot \frac{1}{2} \cdot 2^n$	Definition of negative exponent
$f(n) = \frac{45}{2} \cdot 2^n$	Multiply.

So $g_n = 45 \cdot 2^{n-1}$ written as a function in function notation is $f(n) = \frac{45}{2} \cdot 2^n$.

2. In the previous lesson, you created graphic organizers that identified the explicit formulas for six geometric sequences. Rewrite each explicit formula in function notation.

a. Sequence C

$g_n = -2 \cdot 3^{n-1}$

b. Sequence F

$g_n = 1234 \cdot 0.1^{n-1}$

c. Sequence I

$g_n = 1 \cdot 10^{n-1}$

d. Sequence J

$g_n = -5 \cdot \frac{1}{2}^{n-1}$

e. Sequence M

$g_n = -16 \cdot \left(-\frac{1}{4}\right)^{n-1}$

f. Sequence P

$g_n = -4 \cdot (-3)^{n-1}$

3. Based on the formulas, do the geometric sequences belong to the same function family? Explain your reasoning.

4. What is the relationship between the common ratio of a geometric sequence and the base of the power in an exponential function?

5. What is the relationship between the first term of a geometric sequence and the coefficient of the power in an exponential function?

6. Complete the table by writing each part of the exponential function that corresponds to each part of the geometric sequence.

Geometric Sequence	**Exponential Function**
$g_n = g_1 \cdot r^{n-1}$	$f(x) = a \cdot b^x$
g_n	
g_1	
r	
n	

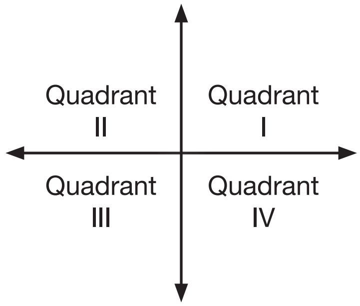
Talk the Talk

1. Complete each statement with always, sometimes, or never. Explain your reasoning for each statement.

   a. An arithmetic sequence can ____________ be represented as a linear function whose domain is the set of natural numbers.

   b. A geometric sequence can ____________ be represented as an exponential function whose domain is the set of natural numbers.

2. Determine whether the statement is true or false. If it is false, explain why it is false. Remember, the coordinate plane is split into four quadrants, as shown.



a. An arithmetic sequence will always begin in Quadrant 1.

b. An arithmetic sequence will never begin in Quadrant 3.

4

c. A geometric sequence will sometimes begin in Quadrant 2.

d. A geometric sequence will always begin in Quadrant 4.

Be prepared to share your solutions and methods.

4

Chapter 4 Summary

KEY TERMS

- sequence (4.1)
- term of a sequence (4.1)
- infinite sequence (4.1)
- finite sequence (4.1)
- arithmetic sequence (4.2)
- common difference (4.2)
- geometric sequence (4.2)
- common ratio (4.2)
- index (4.3)
- explicit formula (4.3)
- recursive formula (4.3)

4.1 Recognizing and Describing Patterns

A sequence is a pattern involving an ordered arrangement of numbers, geometric figures, letters, or other objects. A term in a sequence is an individual number, figure, or letter in the sequence. Many different patterns can generate a sequence of numbers.

Example

Bennie bought a new album for collecting baseball cards. The album can hold 275 baseball cards. At the end of each week, Bennie buys 15 baseball cards with his leftover lunch money from that week.

A sequence to represent how many baseball cards Bennie can fit into his album after 6 weeks is 275 cards, 260 cards, 245 cards, 230 cards, 215 cards, and 200 cards. This sequence begins at 275 and decreases by 15 with each term.

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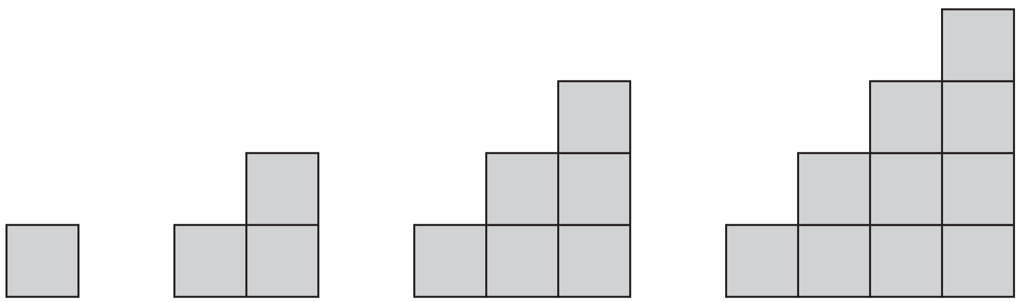
4.1 Identifying Infinite and Finite Sequences

A sequence that continues on forever is called an infinite sequence. A sequence that terminates is called a finite sequence.

Example

The first four terms in this sequence show how many total squares are in each set of steps as a new column is added.



The pattern for this sequence increases forever, therefore it is an infinite sequence.

4.2 Recognizing Arithmetic Sequences and Determining the Common Difference

An arithmetic sequence is a sequence of numbers in which the difference between any two consecutive terms is a constant. This constant is called the common difference and is typically represented by the variable *d*. The common difference of a sequence is positive if the same positive number is added to each term to produce the next term. The common difference of a sequence is negative if the same negative number is added to each term to produce the next term.

Example

Consider the sequence shown.

$$14,\ 16\frac{1}{2},\ 19,\ 21\frac{1}{2},\ \ldots$$

The pattern of this sequence is to add $2\frac{1}{2}$ to each term to produce the next term. This is an arithmetic sequence, and the common difference d is $2\frac{1}{2}$.

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4.2 Recognizing Geometric Sequences and Determining the Common Ratio

A geometric sequence is a sequence of numbers in which the ratio between any two consecutive terms is a constant. The constant, which is either an integer or a fraction, is called the common ratio and is typically represented by the variable *r*.

Example

Consider the sequence shown. The common ratio is $\frac{1}{3}$.

$$27,\ 9,\ 3,\ 1,\ \frac{1}{3},\ \frac{1}{9}$$

The pattern is to multiply each term by the same number, $\frac{1}{3}$, to determine the next term. Therefore, this sequence is geometric, and the common ratio r is $\frac{1}{3}$.

4.3 Writing Explicit Formulas for Arithmetic and Geometric Sequences

An explicit formula for a sequence is a formula for calculating each term of the sequence using the index, which is a term's position in the sequence. The explicit formula for determining the nth term of an arithmetic sequence is $a_n = a_1 + d(n - 1)$. The explicit formula for determining the nth term of a geometric sequence is $g_n = g_1 \cdot r^{n-1}$.

Example

Tom just bought a new cactus plant for his office. The cactus is currently 3 inches tall and will grow $\frac{1}{4}$ inch every month. The explicit formula for arithmetic sequences can be used to determine how tall the cactus will be in 12 months.

$$a_n = a_1 + d(n - 1)$$

$$a_{12} = 3 + \frac{1}{4}(12 - 1)$$

$$a_{12} = 3 + \frac{1}{4}(11)$$

$$a_{12} = 5\frac{3}{4}$$

In 12 months, the cactus will be $5\frac{3}{4}$ inches tall.

4.3 Writing Recursive Formulas for Arithmetic and Geometric Sequences

A recursive formula expresses each new term of a sequence based on a preceding term of the sequence. The recursive formula for determining the nth term of an arithmetic sequence is $a_n = a_{n-1} + d$. The recursive formula for determining the nth term of a geometric sequence is $g_n = g_{n-1} \cdot r$. When using the recursive formula, it is not necessary to know the first term of the sequence.

Example

Consider the geometric sequence shown. The common ratio is $\frac{1}{4}$.

$$32, 8, 2, \frac{1}{2}, \ldots$$

The 5th term of the sequence can be determined by using the recursive formula.

$$g_n = g_{n-1} \cdot r$$

$$g_5 = g_4 \cdot r$$

$$g_5 = \frac{1}{2} \cdot \frac{1}{4}$$

$$g_5 = \frac{1}{8}$$

The 5th term of the sequence is $\frac{1}{8}$.

4.4

Graphing Arithmetic and Geometric Sequences

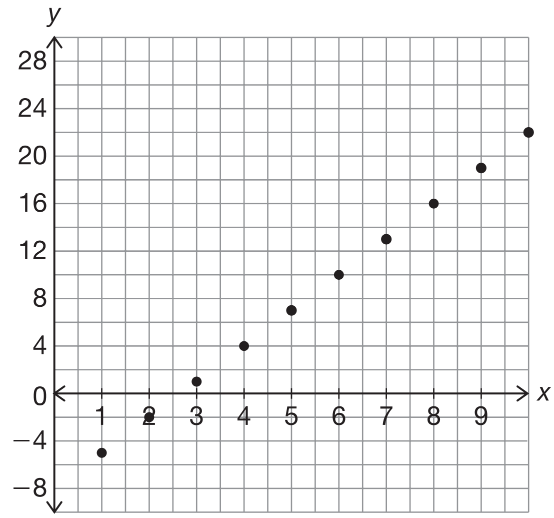
Graphing the terms of a sequence on a coordinate plane is one strategy for organizing values in a sequence. The graphical behavior of the points can help determine the type of sequence that is being graphed. All arithmetic sequences are linear, but not all geometric sequences have the same shape.

Example

Consider the explicit formula for sequence *a*.

$a_n = -5 + 3(n - 1)$

Term Number (*n*)	Term Value
1	−5
2	−2
3	1
4	4
5	7
6	10
7	13
8	16
9	19
10	22



Because sequence *a* is an arithmetic sequence, the graph is in the shape of a line. The graph is discrete because the terms are integer values beginning at 1.

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4.5 Writing an Arithmetic Sequence as a Linear Function

Arithmetic sequences belong to the linear function family because both arithmetic sequences and linear functions have a constant rate of change. Arithmetic sequences and linear functions are both in the form $f(x) = mx + b$, and the common difference of an arithmetic sequence is the slope of a linear function.

Example

An explicit formula for an arithmetic sequence can be written using function notation.

$$a_n = -12 + 3\frac{1}{2}(n - 1)$$

$$f(n) = -12 + 3\frac{1}{2}(n - 1)$$

$$f(n) = -12 + 3\frac{1}{2}n - 3\frac{1}{2}$$

$$f(n) = 3\frac{1}{2}n - 12 - 3\frac{1}{2}$$

$$f(n) = 3\frac{1}{2}n - 15\frac{1}{2}$$

4.5 Writing a Geometric Sequence as an Exponential Function

Not all geometric sequences belong to the same function family. Geometric sequences with positive common ratios belong in the exponential function family. The common ratio of a geometric sequence is the base of an exponential function.

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Example

An explicit formula for a geometric sequence can be written using function notation.

$$g_n = 12 \cdot 4^{n-1}$$

$$f(n) = 12 \cdot 4^n \cdot 4^{-1}$$

$$f(n) = 12 \cdot 4^{-1} \cdot 4^n$$

$$f(n) = 12 \cdot \frac{1}{4} \cdot 4^n$$

$$f(n) = 3 \cdot 4^n$$

This geometric sequence belongs in the exponential function family because the common ratio (or base) is positive.

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