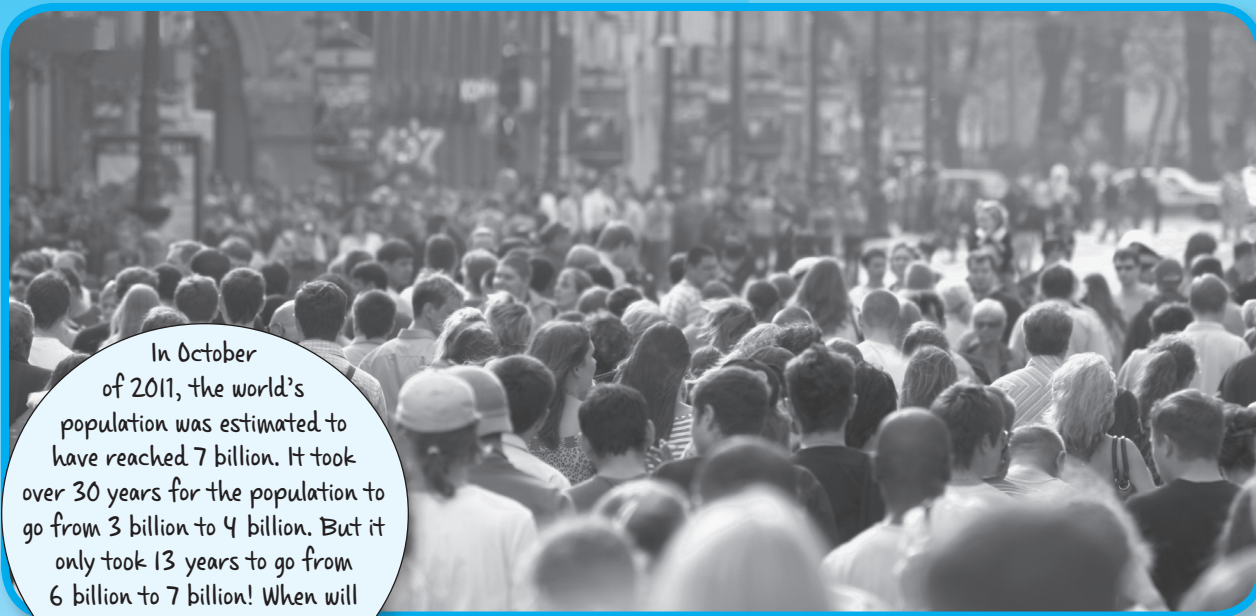


Exponential Functions

5



In October of 2011, the world's population was estimated to have reached 7 billion. It took over 30 years for the population to go from 3 billion to 4 billion. But it only took 13 years to go from 6 billion to 7 billion! When will we hit 8 billion?



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Go for the Curve!

Comparing Linear and Exponential Functions

LEARNING GOALS

In this lesson, you will:

- Construct and identify linear and exponential functions from sequences.
- Compare graphs, tables, and equations of linear and exponential functions.
- Construct a linear function from an arithmetic sequence.
- Construct an exponential function from a geometric sequence.
- Compare formulas for simple interest and compound interest.

KEY TERMS

- simple interest
- compound interest

It might seem unfair, but many banks will charge you money for not having money. And they'll pay you money if you have a lot of it. In 2008, U.S. banks made about \$24 billion in overdraft fees—a 35% increase from 2006. By contrast, personal interest income in 2008 was over \$1 trillion!

When you deposit money in a bank account which accrues interest, your money doesn't just sit there, waiting for you to withdraw it. The bank lends this money to people who want to buy cars, houses, and pay for college.

Banks collect interest on these loans and reward you for your contribution. The more money you have in an interest-earning account, the more you are rewarded!

PROBLEM 1 Let's Build a Formula for Simple Interest



Nico considers depositing money into an account that earns *simple interest* each year.

In a **simple interest** account, the interest earned at the end of each year is a percent of the original deposited amount (also known as the original principal). If Nico deposits \$500 into an account that earns 2% simple interest each year, 2% of the original principal, or $\$500(0.02)$, is added to the account each year.

The balance after 1 year would be $\$500 + \$500(0.02)$, or \$510. The balance after 2 years would be $\$510 + \$500(0.02)$, or \$520, and so on.

Remember that when you are given a percent, you can convert it to a decimal before you perform any calculations.



- Suppose that Nico deposits \$1000 into an account that earns 5% simple interest each year. Complete the table to show Nico's account balance after each year. Show your work. The first three rows have been completed for you.

Value in the time column is 1 less than the corresponding term number n . This is because a sequence always begins with the first term at $n = 1$, but the problem situation begins at time = 0.

Term Number (n)	Time (years)	Interest Earned (dollars)	Account Balance (dollars)
1	0	0	1000
2	1	$1000(0.05) = 50$	$1000 + 1000(0.05) = 1050$
3	2	$1000(0.05) = 50$	$1050 + 1000(0.05) = 1100$
4			
5			
6			

5



- Is this problem situation a sequence? If so, describe the sequence type and how you know. If it is not a sequence, explain why not.



- Suppose P_t represents the account balance after t years. So, P_0 represents the account balance after 0 years (the original principal), P_1 represents the account balance after 1 year, P_2 represents the account balance after 2 years, and so on. If r represents the interest rate, what does $P_0 \cdot r$ represent?

The amount in the account after 1 year, P_1 , is equal to the initial amount plus the interest earned.

$$\begin{array}{c} \text{Amount after} \\ \text{1 year} \end{array} = \begin{array}{c} \text{Initial} \\ \text{amount} \end{array} + \begin{array}{c} \text{Interest} \\ \text{earned} \end{array}$$

$$P_1 = P_0 + P_0 \cdot r$$

This formula represents the balance in the account after 1 year in terms of P_0 .



You can use the formula for P_1 to determine the formula for P_2 in terms of P_0 . The amount in the account after 2 years, P_2 , is equal to the amount in the account after 1 year plus the interest earned.

To write the formula for P_2 in terms of P_0 , substitute the equivalent expression for P_1 .

$$\begin{array}{c} \text{Amount after} \\ \text{2 years} \end{array} = \begin{array}{c} \text{Amount} \\ \text{after 1 year} \end{array} + \begin{array}{c} \text{Interest} \\ \text{earned} \end{array}$$

$$P_2 = P_1 + P_0 \cdot r$$

$$P_2 = [P_0 + P_0 \cdot r] + P_0 \cdot r$$



You can continue this sequence to determine the balance in the account after 3 years, 4 years, and so on, in terms of P_0 .



4. Write a formula in terms of P_0 to represent the balance in the account after:

a. 3 years.

$$P_3 = P_2 + P_0 \cdot r$$

$$P_3 = \underline{\hspace{2cm}} + \underline{\hspace{2cm}}$$

b. 4 years.

$$P_4 = P_3 + P_0 \cdot r$$

$$P_4 = \underline{\hspace{2cm}} + \underline{\hspace{2cm}}$$

c. 5 years.

$$P_5 = \underline{\hspace{2cm}} + \underline{\hspace{2cm}}$$

For this exercise, don't simplify! And, remember to keep each formula in terms of P_0 and r , just like in the worked example.



5. Is this an arithmetic or geometric sequence? How do you know? Determine the common difference or common ratio from the formulas in Question 4.
6. Substitute the original principal P_0 , and the common difference or common ratio into the explicit formula for the sequence.
7. Remember that t represents the time in years. Since t begins at 0 and the term number n begins at 1, t is always 1 less than n . How can you represent t algebraically in terms of n ?
 $t =$ _____

8. Let $P(t)$ represent the account balance after t years. Use your answers in Questions 6 and 7 to write the formula for the balance $P(t)$ in the simple interest account as a function of time t .

Recall that an arithmetic sequence can be written as a function in function notation.



9. Write Nico's balance in the simple interest account described in Question 1 as a function of time t . Recall that the original principal is \$1000 and the interest rate is 5%. Use the function to verify the values in your table.

5



10. Determine which function family represents the simple interest formula. Explain your reasoning.



11. Use the function you wrote in Question 9 to determine Nico's account balance after:

a. 8 years.

b. 100 years.

12. Use the **intersection** feature of a graphing calculator to determine the number of years it will take for the balance in Nico's account to:

a. reach \$1600.



b. double.

PROBLEM 2 Now Let's Build a Formula for Compound Interest



Raul considers depositing money into an account that earns *compound interest* each year.

In a **compound interest** account, the interest earned at the end of each year is a percent of the account balance at the beginning of the year. For example, if \$500 is deposited into an account that earns 2% compound interest each year, the balance after 1 year would be $\$500 + \$500(0.02) = \$510$, the balance after 2 years would be $\$510 + \$510(0.02) = \$520.20$, and so on.



1. Suppose that Raul deposits \$1000 into an account that earns 5% compound interest each year. Complete the table to show Raul's account balance after each year. Show your work. The first three rows have been completed for you.

Term Number (n)	Time (years)	Interest Earned (dollars)	Account Balance (dollars)
1	0	0	1000
2	1	$1000(0.05) = \$50$	$1000 + 1000(0.05) = 1050$
3	2	$1050(0.05) = 52.5$	$1050 + 1050(0.05) = 1102.5$
4			
5			
6			



5



Suppose that P_0 represents Raul's original principal, and r represents the interest rate in the compound interest account. So, the formula

$$P_1 = P_0 + P_0 \cdot r$$

represents the balance in the account in terms of P_0 after 1 year. You can use the Distributive Property to rewrite this formula as

$$P_1 = P_0(1 + r)$$



2. Use the Distributive Property to write a formula to represent the balance in the account in terms of P_0 after the given number of years. The first one has been done for you.

- a. 2 years

$$P_2 = P_1 + P_1 \cdot r$$

$$P_2 = P_1(1 + r)$$

$$P_2 = [P_0(1 + r)](1 + r) = P_0(1 + r)(1 + r)$$

- b. 3 years

$$P_3 = P_2 + P_2 \cdot r$$

$$P_3 =$$

$$P_3 =$$

- c. 4 years

$$P_4 = P_3 + P_3 \cdot r$$

$$P_4 =$$

$$P_4 =$$



3. Is this an arithmetic or geometric sequence? How do you know? Determine the common difference or common ratio from the formulas in Question 2.



4. Substitute Raul's original principal, P_0 , and the common difference or common ratio into the explicit formula for the sequence.

5

5. Let $P(t)$ represent Raul's account balance after t years. Remember that time t is 1 less than the term number n . Write the formula for the balance $P(t)$ in the compound interest account as a function of time t .

6. Write the balance in Raul's compound interest account as a function of time t . Recall that the original principal is \$1000 and the interest rate is 5%. Use the function to verify the values in your table.



7. Determine which function family represents the compound interest formula. Explain your reasoning.



8. Use the function you wrote in Question 6 to determine the account balance after:
- 8 years.
 - 100 years.



9. Use the **intersection** feature of a graphing calculator to determine the number of years it will take for the balance in Raul's account to:
- reach \$1600.
 - double.

Talk the Talk

5



You now know the formulas for simple and compound interest.

The simple interest formula is

$$\begin{aligned} P_t &= P_0 + (P_0 \cdot r)t \\ &= P_0 (1 + rt) \end{aligned}$$

where P_t represents the balance in the account after t years, P_0 represents the original principal, and r represents the interest rate each year (written as a decimal).

The compound interest formula is

$$P_t = P_0 \cdot (1 + r)^t$$

where P_t represents the balance in the account after t years, P_0 represents the original principal, and r represents the interest rate each year (written as a decimal).



1. Use the simple and compound interest formulas from the situations for Nico's simple interest account and Raul's compound interest account to complete the table. Round the values to the nearest cent.

Quantity	Time	Simple Interest Balance	Compound Interest Balance
Units			
Expression			
	0		
	1		
	2		
	8		
	100		

2. Terrell is looking for some financial advice. He has the option to deposit \$1000 into the simple interest account just like Nico's account, or a compound interest account just like Raul's account. The compound interest account would cost him a one-time start-up fee of \$200. The simple interest account is free. Into which account would you tell Terrell to put his money and why?

3. Graph the simple interest and compound interest functions on your calculator. Then, sketch the graphs on the given grid. Use the bounds $[0, 40] \times [0, 6000]$.



4. What is the average rate of change for the simple interest function? Explain how you know.
5. Determine the average rate of change between each pair of values given for the compound interest function.
- Between $t = 0$ and $t = 1$:

 - Between $t = 1$ and $t = 2$:

 - Between $t = 2$ and $t = 8$:

 - Between $t = 8$ and $t = 100$:
6. Compare the average rates of change for the simple and compound interest accounts.
- a. What do you notice?

 - b. What does this tell you about the graphs of linear and exponential functions?

5



Be prepared to share your solutions and methods.

Downtown and Uptown

Graphs of Exponential Functions

LEARNING GOALS

In this lesson, you will:

- Solve exponential functions using the intersection of graphs.
- Analyze asymptotes of exponential functions and their meanings in context.
- Identify the domain and range of exponential functions.
- Analyze and graph decreasing exponential functions.
- Compare graphs of linear and exponential functions through intercepts, asymptotes, and end behavior.

KEY TERM

- horizontal asymptote

As the world economy has shifted from agriculture to industry over the past century, more and more people live in cities. This trend is known as urbanization.

People tend to move to cities for career opportunities and more cultural interactions. In 1900, only 13% of the world's population lived in urban environments. By 2005, this figure rose to 49%. Some experts have predicted that by 2030, 60% of the global population will consist of city dwellers.

One negative effect of urbanization can be to drive prices dramatically higher. In Manila, one of the densest cities in the world, skyrocketing prices have forced many people to live in slums.

PROBLEM 1 Downtown and Uptown



At this moment, the population of Downtown is 20,000, and the population of Uptown is 6000. But over many years, people have been moving away from Downtown at a rate of 1.5% every year. At the same time, Uptown's population has been growing at a rate of 1.8% each year.

1. What are the independent and dependent quantities in each situation?
2. Which city's population can be represented as an increasing function, and which can be represented as a decreasing function?

Let's analyze the population growth of Uptown. In 1 year from now, the population of Uptown will be

$$6000 + 6000(0.018) = 6108.$$

The population will be 6108 people in Uptown 1 year from now.



3. Write and simplify an expression that represents the population of Uptown:
 - a. 2 years from now.

- b. 3 years from now.



4. How can you tell that this function is an exponential function? Explain your reasoning.

5



You can use the formula for compound interest to determine the function for Uptown's increasing population. Recall that the formula for compound interest is $P(t) = P(1 + r)^t$, where $P(t)$ represents the amount in the account after a certain amount of time in years, r is the interest rate written as a decimal, and t is the time in years.

5. In the compound interest formula, substitute Uptown's starting population for P and the rate of population growth for r .
 - a. Write the function, $U(t)$, showing Uptown's population growth as a function of time in years.
 - b. Use your answers to Question 3 and a calculator to verify that your function is correct.

Now let's analyze the population decline of Downtown.



6. Write and simplify an expression that represents the population of Downtown. The first one has been done for you.

- a. 1 year from now.

$$20,000 - 20,000(0.015) = 19,700$$

The population of Downtown will be 19,700 people
1 year from now.

- b. 2 years from now.

- c. 3 years from now.

7. Rewrite the expressions for the population decline in Downtown using the Distributive Property. The first one has been done for you.

- a. 1 year from now.

$$20,000 - 20,000(0.015)$$

$$20,000(1 - 0.015)$$

- b. 2 years from now.

- c. 3 years from now.

8. Use the compound interest formula and your expressions in Question 7 to write the function, $D(t)$, showing Downtown's population decline as a function of time in years.

9. Think about each function as representing a sequence. What is the common ratio in simplest form, or the number that is multiplied each time to get the next term, in each sequence?

Because the population is declining, you have to subtract the change in population each year.



5

What happens when the common ratio is 1?





10. Explain how the common ratios determine whether the exponential functions for the change in population are increasing or decreasing.

PROBLEM 2 Graphing, Finally!



Let's examine the properties of the graphs of the functions for Downtown and Uptown. Here are the functions again:

$$\text{Downtown: } D(t) = 20,000(1 - 0.015)^t \quad \text{Uptown: } U(t) = 6000(1 + 0.018)^t$$

1. Use a graphing calculator to graph both functions using the bounds $[-100, 100] \times [0, 30,000]$.
2. Let's analyze the y -intercepts of each function.
 - a. Identify the y -intercepts.
 - b. Interpret the meaning of the y -intercept in terms of this problem situation.
 - c. Describe how you can determine the y -intercept of each function using just the formula for population increase or decrease.
3. Use a graphing calculator to answer each question. Describe your strategy.
 - a. How long will it take for Downtown's population to be half of what it is now?
 - b. How long will it take for Uptown's population to double from what it is now?

- c. How many years from now will the populations of Downtown and Uptown be equal? Determine the approximate populations.

4. Use the **TRACE** function on your calculator to determine when the population of Downtown was 0. Then, determine when the population of Uptown was 0. What do you notice?



5. Determine the point at which each function is equal to 0 by graphing the line $y = 0$ and determining the intersection point of this line with each function. What do you notice?



Each population function you graphed has a *horizontal asymptote*. A **horizontal asymptote** is a horizontal line that a function gets closer and closer to, but never intersects.

6. Write the equation for the horizontal asymptote of each population function.

7. Does the horizontal asymptote make sense in terms of this problem situation? Explain your reasoning.

8. Identify the domain and range of each function.

PROBLEM 3 The Multiple Representations of Exponentials



1. Complete the table and sketch a graph for each exponential function of the form $f(x) = ab^x$. Then determine the x -intercept(s), y -intercept, asymptote, domain, range, and interval(s) of increase/decrease.

a. $f(x) = 3^x$

x	$f(x)$
-2	
-1	
0	
1	
2	
3	



x -intercept(s):
 y -intercept:
 asymptote:
 domain:
 range:
 interval(s) of increase/decrease:

Analyze the form of each function and make a prediction about the shape of the graph before you start. What do the a and b values tell you?



b. $g(x) = \left(\frac{1}{2}\right)^x$

x	$g(x)$
-2	
-1	
0	
1	
2	
3	



x -intercept(s):
 y -intercept:
 asymptote:
 domain:
 range:
 interval(s) of increase/decrease:

c. $k(x) = 5 \cdot 2^x$

x	$k(x)$
-2	
-1	
0	
1	
2	
3	



x -intercept(s):

y -intercept:

asymptote:

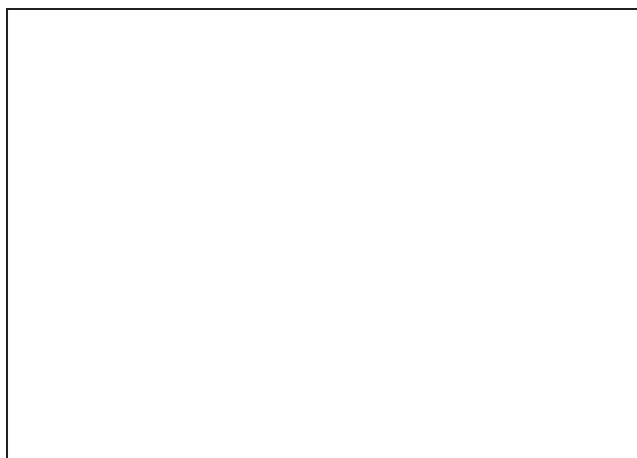
domain:

range:

interval(s) of increase/decrease:

d. $p(x) = -4^x$

x	$p(x)$
-2	
-1	
0	
1	
2	
3	



x -intercept(s):

y -intercept:

asymptote:

domain:

range:

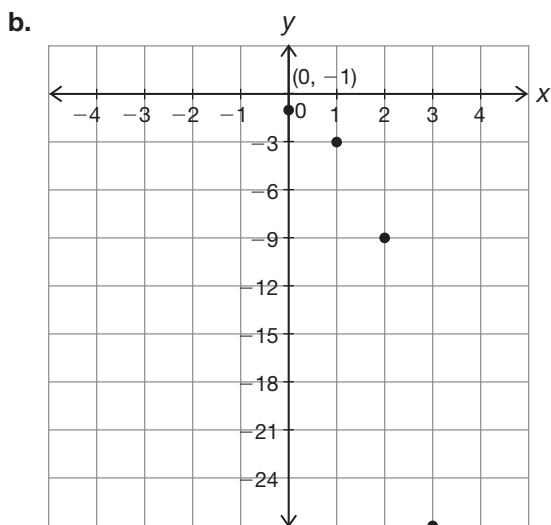
interval(s) of increase/decrease:

2. Write an exponential equation of the form $y = ab^x$ for each. Explain your reasoning.

a.

x	y
0	1
1	4
2	16
3	64

If you know the y -intercept, then you know the a -value. You can use any of the other ordered pairs to determine the b -values.



3. Given a function of the form $f(x) = ab^x$.

a. What does the a -value tell you?

b. What does the b -value tell you?



Be prepared to share your solutions and methods.

Let the Transformations Begin!

5.3

Translations of Linear and Exponential Functions

LEARNING GOALS

In this lesson, you will:

- Translate linear and exponential functions vertically.
- Translate linear and exponential functions horizontally.

KEY TERMS

- basic function
- transformation
- vertical translation
- coordinate notation
- argument of a function
- horizontal translation

Computer programmers write and use functions in much the same way as you do in mathematics. For example, a programmer may write the function **fabs(x)** in a programming language. This function calculates the absolute value of the variable x .

The name of the function is “fabs,” and the variable inside the parentheses is known as the argument. Input is “passed” (either from the program itself or from a user of the program) to the argument, and the function “returns” an output—in this case, the absolute value of x .

How is this programming function related to the functions you have been studying?

PROBLEM 1 Vertical Translations



Consider the three linear functions shown.

- $g(x) = x$
- $c(x) = (x) + 3$
- $d(x) = (x) - 3$

The first function is the *basic function*. A **basic function** is the simplest function of its type. In this case, $g(x) = x$ is the simplest linear function. It is in the form $f(x) = ax + b$, where $a = 1$ and $b = 0$.

You can write the given functions $c(x)$ and $d(x)$ in terms of the basic function $g(x)$. For example, because $g(x) = x$, you can substitute $g(x)$ for x in the equation for $c(x)$, as shown.

$$\begin{array}{l} c(x) = (x) + 3 \\ \quad \quad \downarrow \\ c(x) = g(x) + 3 \end{array}$$



1. Write the function $d(x)$ in terms of the basic function $g(x)$.

$$d(x) = \underline{\hspace{2cm}}$$

2. Describe the operation performed on the basic function $g(x)$ to result in each of the equations for $c(x)$ and $d(x)$.

3. Use a graphing calculator to graph each function with the bounds $[-10, 10] \times [-10, 10]$. Then, sketch the graph of each function. Label each graph.



4. Compare the y -intercepts of the graphs of $c(x)$ and $d(x)$ to the y -intercept of the basic function $g(x)$. What do you notice?
5. Write the y -value of each ordered pair for the three given functions.

$g(x) = x$	$c(x) = (x) + 3$	$d(x) = (x) - 3$
$(-2, \underline{\quad})$	$(-2, \underline{\quad})$	$(-2, \underline{\quad})$
$(-1, \underline{\quad})$	$(-1, \underline{\quad})$	$(-1, \underline{\quad})$
$(0, \underline{\quad})$	$(0, \underline{\quad})$	$(0, \underline{\quad})$
$(1, \underline{\quad})$	$(1, \underline{\quad})$	$(1, \underline{\quad})$
$(2, \underline{\quad})$	$(2, \underline{\quad})$	$(2, \underline{\quad})$



6. Use the table to compare the ordered pairs of the graphs of $c(x)$ and $d(x)$ to the ordered pairs of the graph of the basic function $g(x)$. What do you notice?



A **transformation** is the mapping, or movement, of all the points of a figure in a plane according to a common operation. A **vertical translation** is a type of transformation that shifts the entire graph up or down. A vertical translation affects the y -coordinate of each point on the graph.

Coordinate notation is a notation that uses ordered pairs to describe a transformation in a coordinate plane. For example, you can use the coordinate notation shown to indicate a vertical translation.

$$(x, y) \rightarrow (x, y + b), \text{ where } b \text{ is a real number.}$$

7. Use coordinate notation to represent the vertical translation of each function.

- $g(x) = x$
 (x, y)

- $c(x) = (x) + 3$
 $(x, y) \rightarrow \underline{\hspace{2cm}}$

- $d(x) = (x) - 3$
 $(x, y) \rightarrow \underline{\hspace{2cm}}$



Now, let's consider the three exponential functions shown.

- $h(x) = 2^x$
- $s(x) = (2^x) + 3$
- $t(x) = (2^x) - 3$

In this case, $h(x) = 2^x$ is the basic function because it is the simplest exponential function with a base of 2. It is in the form $f(x) = a \cdot b^x$, where $a = 1$ and $b = 2$.

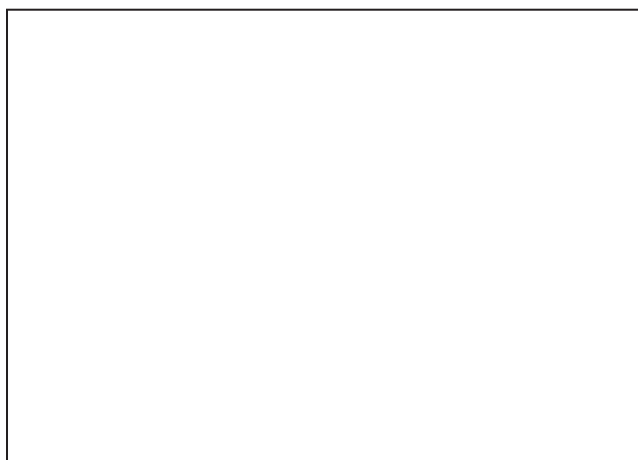


8. Write the functions $s(x)$ and $t(x)$ in terms of the basic function $h(x)$. Then, describe the operation performed on the basic function $h(x)$ to result in each of the equations for $s(x)$ and $t(x)$.

$$s(x) = \underline{\hspace{4cm}}$$

$$t(x) = \underline{\hspace{4cm}}$$

9. Use a graphing calculator to graph each function with the bounds $[-10, 10] \times [-10, 10]$. Then, sketch the graph of each function. Label each graph.



10. Compare the y -intercepts of the graphs of $s(x)$ and $t(x)$ to the y -intercept of the graph of the basic function $h(x)$. What do you notice? Are the results the same as when you compared the graphs of the linear functions in Question 4?

11. Write the y -value of each ordered pair for the three given functions.

$h(x) = 2^x$	$s(x) = (2^x) + 3$	$t(x) = (2^x) - 3$
$(-2, \underline{\quad})$	$(-2, \underline{\quad})$	$(-2, \underline{\quad\quad})$
$(-1, \underline{\quad})$	$(-1, \underline{\quad})$	$(-1, \underline{\quad\quad})$
$(0, \underline{\quad\quad})$	$(0, \underline{\quad\quad})$	$(0, \underline{\quad\quad})$
$(1, \underline{\quad\quad})$	$(1, \underline{\quad\quad})$	$(1, \underline{\quad\quad})$
$(2, \underline{\quad\quad})$	$(2, \underline{\quad\quad})$	$(2, \underline{\quad\quad})$

12. Use the table to compare the ordered pairs of the graphs of $s(x)$ and $t(x)$ to the ordered pairs of the graph of the basic function $h(x)$. What do you notice? Are the results the same as when you compared the y -values for the linear functions in Question 6?

13. Explain how you know that the graphs of $s(x)$ and $t(x)$ are vertical translations of the graph of $h(x)$.

5



14. Use coordinate notation to represent the vertical translation of each function.

- $h(x) = 2^x$
 (x, y)
- $s(x) = (2^x) + 3$
 $(x, y) \rightarrow \underline{\hspace{2cm}}$
- $t(x) = (2^x) - 3$
 $(x, y) \rightarrow \underline{\hspace{2cm}}$



15. Describe each graph in relation to its basic function.

- Compare $f(x) = x + b$ to the basic function $g(x) = x$ for $b > 0$.
- Compare $f(x) = x + b$ to the basic function $g(x) = x$ for $b < 0$.
- Compare $f(x) = b^x + k$ to the basic function $h(x) = b^x$ for $k > 0$.

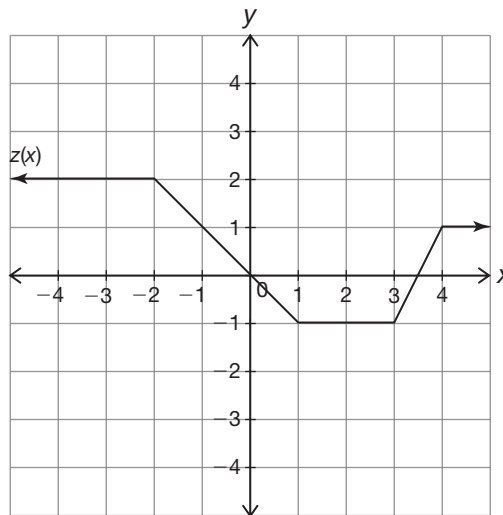


- Compare $f(x) = b^x + k$ to the basic function $h(x) = b^x$ for $k < 0$.



16. The graph of a function $z(x)$ is shown. Sketch the graphs of $z'(x)$ and $z''(x)$.

- $z'(x) = z(x) + 3$
- $z''(x) = z(x) - 4$



Do you remember using a similar notation of prime (') and double prime (") when you translated geometric shapes in middle school?



17. Write the equation of each function after a vertical translation down 6 units.

- $p(x) = 5x$
 $p'(x) = \underline{\hspace{2cm}}$
- $q(x) = 3x^2$
 $q'(x) = \underline{\hspace{2cm}}$
- $r(x) = \frac{1}{2}x^3$
 $r'(x) = \underline{\hspace{2cm}}$

PROBLEM 2 Horizontal Translations



Consider the three exponential functions shown, where $h(x) = 2^x$ is the basic function.

- $h(x) = 2^x$
- $v(x) = 2^{(x+3)}$
- $w(x) = 2^{(x-3)}$

In Problem 1 *Vertical Translations*, the operations that produced the vertical translations were performed on the function $h(x)$. That is, 3 was added to $h(x)$ and 3 was subtracted from $h(x)$. In this problem, the operations are performed on x , which is the *argument* of the function.

The **argument of a function** is the *variable* on which the function operates. So, in this case, 3 is added to x and 3 is subtracted from x .

You can write the given functions $v(x)$ and $w(x)$ in terms of the basic function $h(x)$. To write $v(x)$ in terms of $h(x)$, you just substitute $x + 3$ into the argument for $h(x)$, as shown.

$$h(x) = 2^x$$

$$v(x) = h(x + 3) = 2^{(x+3)}$$

So, $x + 3$ replaces the variable x in the function $h(x) = 2^x$.

1. Write the function $w(x)$ in terms of the basic function $h(x)$.



2. Use a graphing calculator to graph each function with the bounds $[-10, 10] \times [-10, 10]$. Then, sketch the graph of each function. Label each graph.



Sketch the graphs one at a time to help you see which is which.



5

3. Compare the graphs of $v(x)$ and $w(x)$ to the graph of the basic function. What do you notice?

4. Write the x -value of each ordered pair for the three given functions. You can use your graphing calculator to determine the x -values.

$h(x) = 2^x$	$v(x) = 2^{(x+3)}$	$w(x) = 2^{(x-3)}$
(_____, $\frac{1}{4}$)	(_____, $\frac{1}{4}$)	(_____, $\frac{1}{4}$)
(_____, $\frac{1}{2}$)	(_____, $\frac{1}{2}$)	(_____, $\frac{1}{2}$)
(_____, 1)	(_____, 1)	(_____, 1)
(_____, 2)	(_____, 2)	(_____, 2)
(_____, 4)	(_____, 4)	(_____, 4)

Why are there no negative y -values given in this table?
HINT: You learned about it in the previous lesson!



5. Use the table to compare the ordered pairs of the graphs of $v(x)$ and $w(x)$ to the ordered pairs of the graph of the basic function $h(x)$. What do you notice?

A **horizontal translation** of a graph is a shift of the entire graph left or right. A horizontal translation affects the x -coordinate of each point on the graph.

You can use the coordinate notation shown to indicate a horizontal translation.

$$(x, y) \rightarrow (x + a, y), \text{ where } a \text{ is a real number.}$$



6. Use coordinate notation to represent the horizontal translation of each function.

- $h(x) = 2^x$
 (x, y)
- $v(x) = 2^{(x+3)}$
 $(x, y) \rightarrow$ _____
- $w(x) = 2^{(x-3)}$
 $(x, y) \rightarrow$ _____

So, if a constant is added or subtracted **OUTSIDE** a function, like $g(x) + 3$ or $g(x) - 3$, then only the y -values change, resulting in a vertical translation.



And, if a constant is added or subtracted **INSIDE** a function, like $g(x + 3)$ or $g(x - 3)$, then only the x -values change, resulting in a horizontal translation.



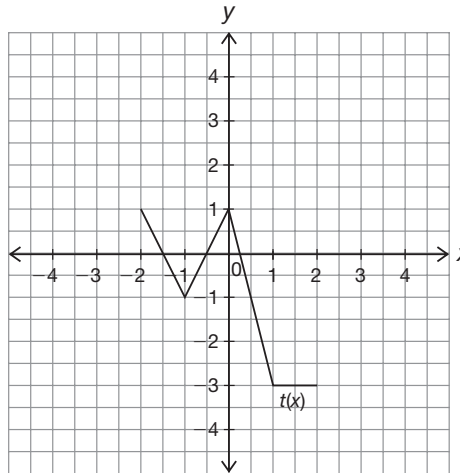
7. Describe each graph in relation to its basic function.
- a. Compare $f(x) = b^{x-c}$ to the basic function $h(x) = b^x$ for $c > 0$.

- b. Compare $f(x) = b^{x-c}$ to the basic function $h(x) = b^x$ for $c < 0$.

8. The graph of a function $t(x)$ is shown. Sketch the graphs of $t'(x)$ and $t''(x)$.

a. $t'(x) = t(x + 3)$

b. $t''(x) = t(x - 1)$



9. Write the equation of each function after a horizontal translation left 10 units.

a. $p(x) = 5x$

$p'(x) =$ _____

b. $q(x) = 3x^2$

$q'(x) =$ _____

c. $r(x) = \frac{1}{2}x^3$

$r'(x) =$ _____

PROBLEM 3 The Curious Case of Linear Functions



Consider the three linear functions shown, where $j(x)$ and $k(x)$ are translations of $g(x)$.

- $g(x) = x$
- $j(x) = g(x) + 5$
- $k(x) = g(x + 5)$

1. Describe the translation of the graph of $g(x)$ that produces $j(x)$. Then, describe the translation of the graph of $g(x)$ that produces $k(x)$.

2. Which function has an operation performed on $g(x)$? Which function has an operation performed on the argument of $g(x)$?

3. Rewrite $j(x)$ so that it is in terms of x instead of in terms of $g(x)$. Then, rewrite $k(x)$ so that it is in terms of x . What do you notice?

5





Kyle, Turk, and Tobias are discussing about whether the translations that produce $j(x)$ and $k(x)$ in Question 1 are the same.

 **Kyle**

The translation that produces $j(x)$ is the same as the translation that produces $k(x)$ because, algebraically, both functions simplify to $x + 5$.

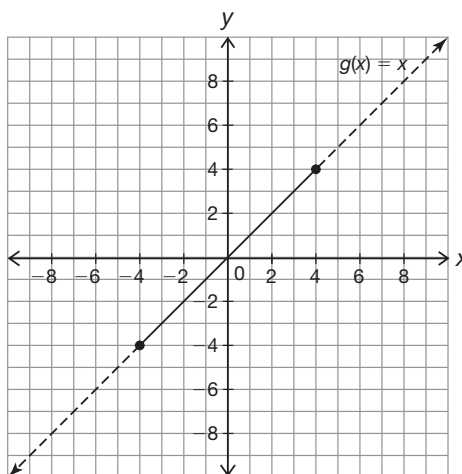
 **Turk**

The translation that produces $j(x)$ is the same as the translation that produces $k(x)$ because when I graph them on my calculator, I get the same line.

 **Tobias**

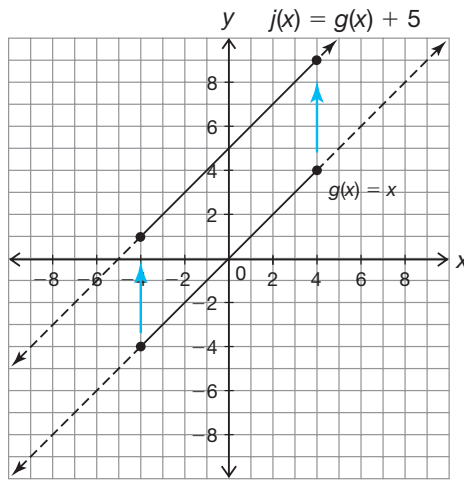
The translation that produces $j(x)$ is not the same as the translation that produces $k(x)$ because $j(x)$ is a vertical translation and $k(x)$ is a horizontal translation.

To see why Tobias is correct, consider a segment of the basic function $g(x) = x$. The function $g(x)$ is shown as a dotted line, and the segment of the function is shown as a solid segment.

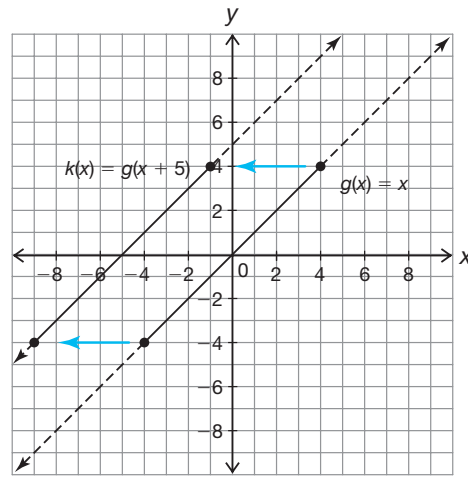


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The function $j(x) = g(x) + 5$ is a vertical translation that is 5 units up from the graph of $g(x)$.



The function $k(x) = g(x + 5)$ is a horizontal translation that is 5 units to the left of the graph of $g(x)$.



Notice that $j(x)$ and $k(x)$ are both part of the same line. However, the segments are in different locations on the coordinate plane.

So, Tobias' reasoning is correct because the translation that produces $j(x)$ is not the same as the translation that produces $k(x)$, even though the algebraic representation and the graphs of the lines are the same.

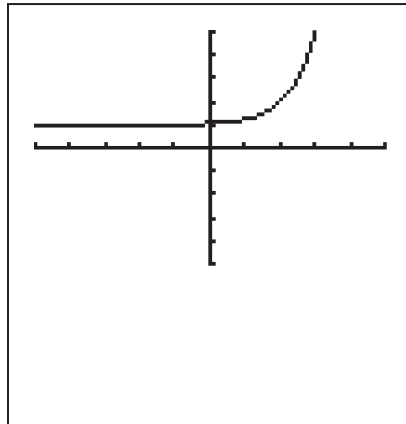
Talk the Talk



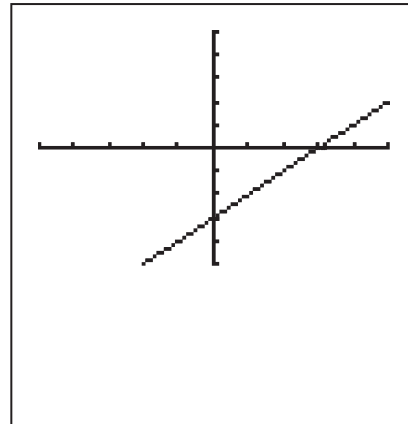
1. Match each function form with its corresponding graph.

- $f(x) = ax + b$, where $a = 1$ and $b > 0$
- $f(x) = ax + b$, where $a = 1$ and $b < 0$
- $f(x) = b^{x-c} + k$, where $c > 0$ and $k > 0$
- $f(x) = b^{x-c} + k$, where $c < 0$ and $k < 0$

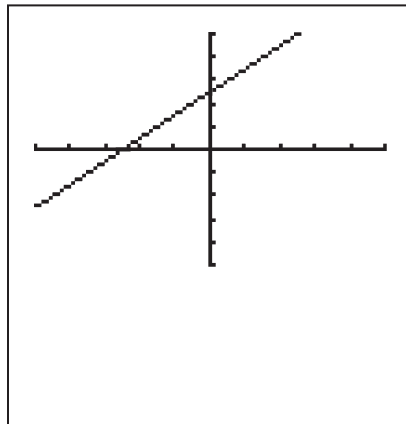
Graph A



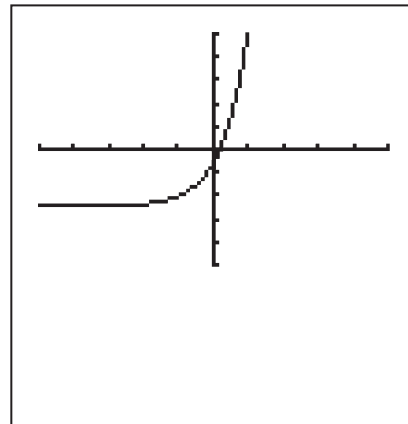
Graph B



Graph C



Graph D



2. Complete the table by describing the graph of each function as a transformation of its basic function.

Function Form	Equation Information	Description of Transformation of Graph
$f(x) = (x) + b$	$b > 0$	
	$b < 0$	
$f(x) = (x - b)$	$b > 0$	
	$b < 0$	
$f(x) = b^x + k$	$b > 1, k > 0$	
	$b > 1, k < 0$	
$f(x) = b^{x-c}$	$b > 1, c > 0$	
	$b > 1, c < 0$	

5



Be prepared to share your solutions and methods.

Take Some Time to Reflect

Reflections of Linear and Exponential Functions

LEARNING GOALS

In this lesson, you will:

- Reflect linear and exponential functions vertically.
- Reflect linear and exponential functions horizontally.
- Determine characteristics of graphs after transformations.

KEY TERMS

- reflection
- line of reflection

You are already familiar with many different types of “reflections” in mathematics. When a negative sign is present, this is a good indication of a reflection.

For example, -5 is a “reflection” of 5 over 0 on the number line.

The power $2^1 = \frac{2}{1}$, but $2^{-1} = \frac{1}{2}$.

In this lesson, you will learn about reflecting functions. Watch out for those negative signs!

PROBLEM 1 Reflections



Consider the three exponential functions shown, where $h(x) = 2^x$ is the basic function.

- $h(x) = 2^x$
- $m(x) = -(2^x)$
- $n(x) = 2^{(-x)}$

1. Write the functions $m(x)$ and $n(x)$ in terms of the basic function $h(x)$.

$$m(x) = \underline{\hspace{2cm}}$$

$$n(x) = \underline{\hspace{2cm}}$$

2. Compare $m(x)$ to $h(x)$. Does an operation performed on $h(x)$ or on the argument of $h(x)$ result in the equation for $m(x)$? What is the operation?

3. Compare $n(x)$ to $h(x)$. Does an operation performed on $h(x)$ or on the argument of $h(x)$ result in the equation for $n(x)$? What is the operation?

4. Use a graphing calculator to graph each function with the bounds $[-10, 10] \times [-10, 10]$. Then, sketch the graph of each function. Label each graph.



Before you press the GRAPH key, make a prediction about the shapes of $m(x)$ and $n(x)$.



5. Compare the graphs of $m(x)$ and $n(x)$ to the graph of the basic function $h(x)$. What do you notice?

6. Complete the table of ordered pairs for the three given functions.

$h(x) = 2^x$	$m(x) = -(2^x)$	$n(x) = 2^{-x}$
$(-2, \frac{1}{4})$	$(-2, \text{---})$	$(\text{---}, \frac{1}{4})$
$(-1, \frac{1}{2})$	$(-1, \text{---})$	$(\text{---}, \frac{1}{2})$
$(0, 1)$	$(0, \text{---})$	$(\text{---}, 1)$
$(1, 2)$	$(1, \text{---})$	$(\text{---}, 2)$
$(2, 4)$	$(2, \text{---})$	$(\text{---}, 4)$



7. Use the table to compare the ordered pairs of the graphs of $m(x)$ and $n(x)$ to the ordered pairs of the graph of the basic function $h(x)$. What do you notice?



A **reflection** of a graph is a mirror image of the graph about a *line of reflection*. A **line of reflection** is the line that the graph is reflected about. A horizontal line of reflection affects the y -coordinates, and a vertical line of reflection affects the x -coordinates.

You can use the coordinate notation shown to indicate a reflection about a horizontal line of reflection.

$$(x, y) \rightarrow (x, -y)$$

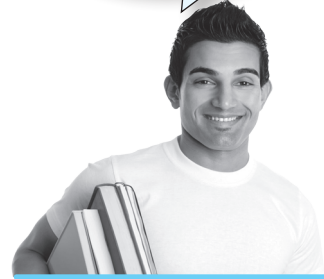
When the negative is on the outside of the function, like $-g(x)$, all the y -values become the opposite of the y -values of $g(x)$. The x -values remain unchanged.



You can also use the coordinate notation shown to indicate a reflection about a vertical line of reflection.

$$(x, y) \rightarrow (-x, y)$$

When the negative is on the inside of the function, like $g(-x)$, all the x -values become the opposite of the x -values of $g(x)$. The y -values remain unchanged.



8. Which function represents a reflection of $h(x)$ over a horizontal line? Which function represents a reflection of $h(x)$ over a vertical line?

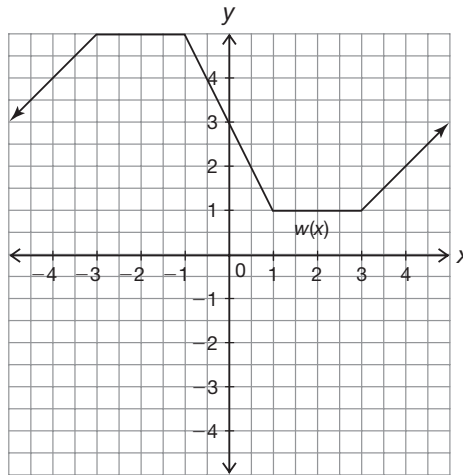
9. Describe each graph in relation to its basic function.
- Compare $f(x) = -(b^x)$ to the basic function $h(x) = b^x$.



- Compare $f(x) = b^{(-x)}$ to the basic function $h(x) = b^x$.



10. The graph of a function $w(x)$ is shown. Sketch the graphs of $w'(x)$ and $w''(x)$.
- $w'(x) = -w(x)$
 - $w''(x) = w(-x)$



11. Write the equation of each function after a reflection about the horizontal line $y = 0$. Then, write the equation after a reflection about the vertical line $x = 0$.
- $a(x) = 5^x$
 Reflection about $y = 0$: $a'(x) = \underline{\hspace{2cm}}$
 Reflection about $x = 0$: $a''(x) = \underline{\hspace{2cm}}$

- b. $b(x) = -2x^2$
 Reflection about $y = 0$: $b'(x) =$ _____
 Reflection about $x = 0$: $b''(x) =$ _____



- c. $c(x) = \frac{5}{4}x^3$
 Reflection about $y = 0$: $c'(x) =$ _____
 Reflection about $x = 0$: $c''(x) =$ _____

PROBLEM 2 Linear Functions . . . Another Curious Case!



Consider the three linear functions shown, where $g(x) = x$ is the basic function.

- $g(x) = x$
- $r(x) = -(x)$
- $s(x) = (-x)$

1. Write the functions $r(x)$ and $s(x)$ in terms of the basic function $g(x)$.

$r(x) =$ _____

$s(x) =$ _____

2. Use a graphing calculator to graph each function with the bounds $[-10, 10] \times [-10, 10]$. Then, sketch the graph of each function. Label each graph.



3. Compare the graphs of $r(x)$ and $s(x)$ to the graph of the basic function $g(x)$. What do you notice?

4. Complete the table of ordered pairs for the three given functions.

$g(x) = x$	$r(x) = -(x)$	$s(x) = (-x)$
$(-2, -2)$	$(-2, \underline{\quad})$	$(\underline{\quad}, -2)$
$(-1, -1)$	$(-1, \underline{\quad})$	$(\underline{\quad}, -1)$
$(0, 0)$	$(0, \underline{\quad})$	$(\underline{\quad}, 0)$
$(1, 1)$	$(1, \underline{\quad})$	$(\underline{\quad}, 1)$
$(2, 2)$	$(2, \underline{\quad})$	$(\underline{\quad}, 2)$

5. Use the table to compare the ordered pairs of the graphs of $r(x)$ and $s(x)$ to the ordered pairs of the graph of the basic function $g(x)$. What do you notice?

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6. Which function represents a reflection of the basic function $g(x)$ over a vertical line?
Which function represents a reflection of the basic function $g(x)$ over a horizontal line?
Why do you think they produce the same graph?

PROBLEM 3**Characteristics of Graphs after Transformations**

1. Use the given characteristics to write an equation and then sketch a graph of $g(x)$.

a. Write an equation and sketch a graph that:

- is an exponential function,
- is continuous,
- is increasing, and
- is translated 2 units to the left of $f(x) = 2^x$.

Equation: $g(x) =$ _____



b. Write an equation and sketch a graph that:

- is an exponential function,
- is continuous,
- is increasing, and
- is translated 5 units down from $f(x) = 2^x$.

Equation: $g(x) =$ _____



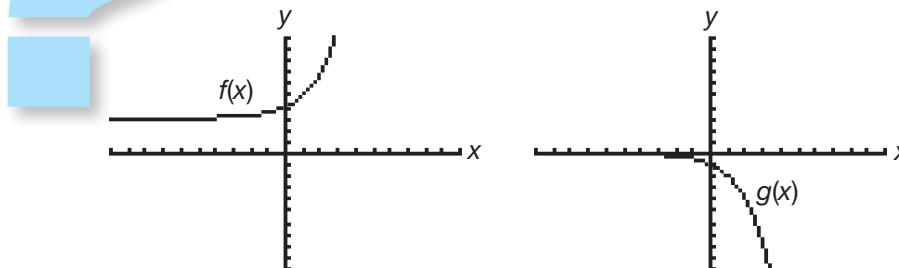
c. Write an equation and sketch a graph that:

- is an exponential function,
- is continuous,
- is decreasing, and
- is a reflection of $f(x) = 2^x$ over the line $x = 0$.

Equation: $g(x) =$ _____



2. Jacob and Kate are comparing the two graphs shown.



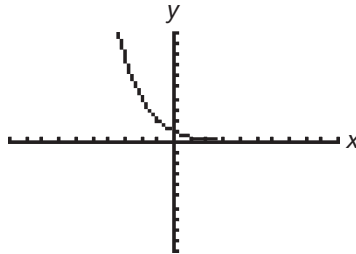
Jacob says that to get the graph of $g(x)$, first translate $f(x)$ down 3 units, and then reflect over the line $y = 0$. Kate says that to get the graph of $f(x)$, first reflect $g(x)$ over the line $y = 0$, and then translate up 3 units.

Who is correct? Explain your reasoning.

3. Choose the transformations in the box performed on $f(x)$ that would produce the graph of $g(x)$. Then sketch the graph of $g(x)$ on the coordinate plane.

vertical translation	horizontal translation
reflection over the line $y = 0$	reflection over the line $x = 0$

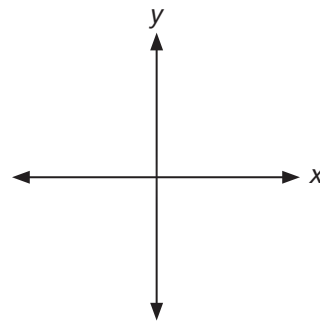
- a. The graph of $f(x)$ is shown.



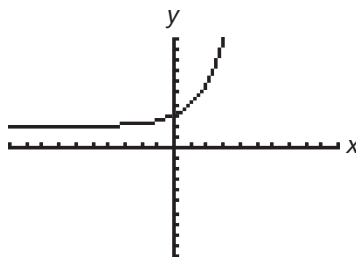
The graph of $g(x)$ has a different y -intercept than the graph of $f(x)$, but the same asymptote as $f(x)$.

Possible transformation(s) on $f(x)$ to produce $g(x)$:

Sketch of $g(x)$



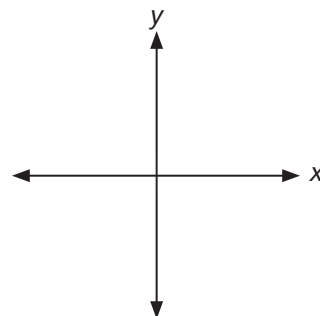
- b. The graph of $f(x)$ is shown.



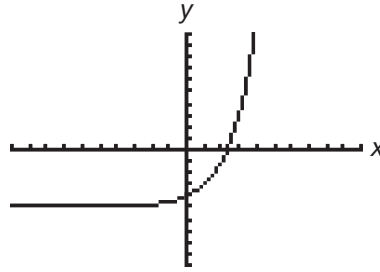
The graph of $g(x)$ has the same y -intercept as $f(x)$, but $g(x)$ is decreasing.

Possible transformation(s) on $f(x)$ to produce $g(x)$:

Sketch of $g(x)$



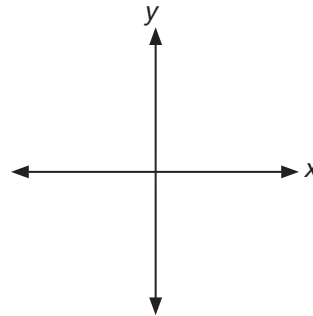
c. The graph of $f(x)$ is shown.



The graph of $g(x)$ has a different asymptote than the graph of $f(x)$, and $g(x)$ is increasing.

Possible transformation(s) on $f(x)$ to produce $g(x)$:

Sketch of $g(x)$



5



Be prepared to share your solutions and methods.

Radical! Because It's Cliché!

Properties of Rational Exponents

LEARNING GOALS

In this lesson, you will:

- Simplify expressions with negative exponents.
- Simplify expressions with rational exponents.
- Write negative powers as positive powers.
- Write rational powers using radicals.
- Find the n th root of a number.
- Write an expression in radical form.

KEY TERMS

- cube root
- index
- n th root
- radicand
- rational exponent

Where does the word “radical” come from? Why did mathematicians choose this word for the symbol $\sqrt{\quad}$?

It turns out that the Latin word “radix” means “root.” The word “radish”—a root vegetable—has the same origin as the word “radical.” When you use radicals in mathematics, you are determining “roots” of numbers.

There are other words, too, that come from *radix*. The word “eradicate” is an example: to uproot or destroy.

PROBLEM 1 Negative Exponents

Three units of time that are smaller than seconds are milliseconds, microseconds, and nanoseconds.



1. Complete each statement to determine:

a. the number of seconds there are in 100 milliseconds.

$$100 \text{ milliseconds} \left(\frac{1 \text{ second}}{1000 \text{ milliseconds}} \right) = \frac{\boxed{} \text{ seconds}}{\boxed{}}$$
$$= \text{_____ second(s)}$$

b. the number of seconds there are in 10 milliseconds.

$$10 \text{ milliseconds} \left(\frac{1 \text{ second}}{1000 \text{ milliseconds}} \right) = \frac{\boxed{} \text{ seconds}}{\boxed{}}$$
$$= \text{_____ second(s)}$$

c. the number of seconds there are in 1 millisecond.

$$1 \text{ millisecond} \left(\frac{1 \text{ second}}{1000 \text{ milliseconds}} \right) = \frac{\boxed{} \text{ second}}{\boxed{}}$$
$$= \text{_____ second(s)}$$

Recall the Quotient Rule of Powers is:

$$\frac{a^m}{a^n} = a^{m-n}, \text{ when } a \neq 0$$

2. Show how you can use powers of 10 to write your answers from Question 1 as a quotient of powers and then as a single power.

a. $\frac{100}{1000} = \frac{\quad}{\quad} = \quad$

b. $\frac{10}{1000} = \frac{\quad}{\quad} = \quad$

c. $\frac{1}{1000} = \frac{\quad}{\quad} = \quad$

When you divide powers with the same base, you keep the base and subtract the exponents.



3. Explain how you can write a power with a negative exponent as a power with a positive exponent.



Remember, any number to the 0 power (except for 0) is 1.



PROBLEM 2 Boat Stability



The weight of a boat depends on its beam width. An equation that relates a boat's weight to its beam width is $w = 64\left(\frac{b}{c}\right)^3$, where w is the boat's weight in pounds, b is the beam width in feet, and c is the capsize factor. When the capsize factor c is less than 2, the boat is less likely to capsize, or turn over.



1. Suppose that a boat has a capsize factor of 3 and a beam width of 6 feet. What is the weight of the boat? Show your work.

5

2. Suppose that a boat has a capsize factor of 3 and a beam width of 9 feet. What is the weight of the boat? Show your work.
3. Explain how you calculated your answers to Questions 1 and 2.
4. Suppose that a boat has a capsize factor of 2 and weighs 1000 pounds.
- Write an equation that you can use to determine the beam width of the boat.

- Transform the equation you wrote to isolate the variable with its power on one side of the equation. Show your work.

Remember, the Quotient to a Power Rule states that

$$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$$



- To determine the value of b , you need to know the number that when multiplied by itself 3 times is equal to 125. What is this number?



- What is the boat's beam width?



You can say that a number b is a **cube root** of a number a if $b^3 = a$.

So, 5 is a cube root of 125 because $5^3 = 125$.

A cube root is indicated by the symbol $\sqrt[3]{\quad}$. The number 3 is called the **index** of the radical. So, $\sqrt[3]{125} = 5$.



5. Complete each statement.

a. $\sqrt[3]{8} = \underline{\hspace{2cm}}$

b. $\sqrt[3]{-27} = \underline{\hspace{2cm}}$

You can extend the idea of square roots and cube roots to solve equations. If n represents a positive number, then a number b is the **n th root** of a if $b^n = a$.

For example, 2 is the 4th root of 16 because $2^4 = 16$.

6. Complete each statement.

a. The number 3 is the 5th root of 243 because $\underline{\hspace{2cm}} = 243$.

b. The number -2 is the cube root of -8 because $(-2)^3 = \underline{\hspace{2cm}}$.

c. The number 4 is the $\underline{\hspace{2cm}}$ root of 4096 because $4^6 = 4096$.

A square root is indicated by the symbol $\sqrt{\quad}$. It is not necessary to write the index of 2 for a square root.



The n th root of a number a is designated as $\sqrt[n]{a}$, where n is the index of the radical and a is the **radicand**, which is the value within the radical.



7. Complete each statement. The first one has been done for you.

a. $\sqrt{100} = \underline{10}$ because $\underline{10^2} = 100$.

b. $\sqrt[3]{216} = \underline{\hspace{2cm}}$ because $\underline{\hspace{2cm}}^3 = 216$.

c. $\sqrt[4]{81} = \underline{\hspace{2cm}}$ because $\underline{\hspace{2cm}}^4 = 81$.

d. $\sqrt[5]{-32} = \underline{\hspace{2cm}}$ because $\underline{\hspace{2cm}}^5 = -32$.





8. Notice that a power can be positive or negative, depending on the base and the exponent.
- When the exponent of a power is an even number, and the base is a positive number, is the value of the power positive or negative? How do you know?
 - When the exponent of a power is an even number, and the base is negative, is the value of the power positive or negative? How do you know?
 - When the exponent of a power is an odd number, and the base is a positive number, is the value of the power positive or negative? How do you know?
 - When the exponent of a power is an odd number, and the base is negative, is the value of the power positive or negative? How do you know?



5

PROBLEM 3 Boat Speed

The hull speed of a boat depends on the length of the hull at the waterline. An equation that relates the speed s in knots and the length, r in feet of the hull at the waterline is $s = 1.34\sqrt{r}$.



- What is the hull speed of a boat that has a length of 16 feet at the waterline? Show your work.
- Write an equation that you can use to determine the length of a boat at its waterline if the boat's hull speed is 6.7 knots.

3. Transform the equation you wrote to isolate the radical on one side of the equation. Show your work.
4. What is the length of the boat at the waterline? How do you know?



5. Complete each step.

$$r = 25$$

Given.

$$r = \underline{\hspace{2cm}}$$

Write 25 as a power.

$$r = \underline{\hspace{2cm}}$$

Substitute \sqrt{r} for 5.



You can write the equation for the length of the boat at the waterline this way:

$$r^1 = (\sqrt{r})^2$$

You know that when you determine the power of a power, you multiply the exponents: $(a^b)^c = a^{bc}$.

Since $(\sqrt{r})^2 = r^1$, \sqrt{r} must be equal to $r^{\frac{1}{2}}$, because $(r^{\frac{1}{2}})^2 = r^{\frac{1}{2}(2)} = r^1$.

A **rational exponent** is an exponent that is a rational number. You can write each n th root using a rational exponent. If n is an integer greater than 1, then $\sqrt[n]{a} = a^{\frac{1}{n}}$.

5



6. Write each radical as a power.

a. $\sqrt[3]{7}$

b. $\sqrt[5]{x}$

c. \sqrt{y}

7. Write each power as a radical.

a. $8^{\frac{1}{4}}$

b. $z^{\frac{1}{5}}$

c. $m^{\frac{1}{3}}$



8. Write the equation for the hull speed of a boat using a rational exponent.



Another equation can be used to determine the speed of a sailboat in knots. This equation is $s = \frac{a}{d^{\frac{2}{3}}}$, where a represents the sail area and d is the boat's displacement.



You can write the equation $s = \frac{a}{d^{\frac{2}{3}}}$ in radical form by using the properties you know about powers:



Given equation. $s = \frac{a}{d^{\frac{2}{3}}}$



Write the power as a product using a unit fraction. $s = \frac{a}{d^{\frac{1}{3} \cdot 2}}$



Use the power of a power rule and the definition of a rational exponent to write the power as a radical. $s = \frac{a}{(\sqrt[3]{d})^2}$



9. Use the equation to calculate the speed s of a boat that has a sail area of 600 square feet and a displacement of 125 cubic feet. Show your work.

10. Write each expression in radical form. Show your work and simplify your answer, if possible.

a. $4^{\frac{3}{2}}$

b. $5^{\frac{3}{4}}$

c. $x^{\frac{4}{5}}$

d. $y^{\frac{2}{3}}$

11. Write each expression in rational exponent form. Show your work and simplify your answer, if possible.

a. $(\sqrt[4]{2})^3$

b. $(\sqrt{5})^4$

c. $(\sqrt[5]{x})^8$

d. $(\sqrt[5]{y})^{10}$



Be prepared to share your solutions and methods.

Checkmate!

Solving Exponential Functions

LEARNING GOALS

In this lesson, you will:

- Use multiple representations to model exponential functions.
- Understand the properties of exponent expressions with positive and negative exponents.
- Solve exponential functions graphically and algebraically using common bases and properties of exponents.
- Investigate increasing and decreasing exponential functions.
- Model inequalities in exponential situations.
- Use technology to graph, analyze, and solve exponential functions.

How do scientists measure the intensity of earthquakes? You may know that those scientists who study earthquakes—seismologists—refer to a scale known as a Richter scale when reporting the strength of an earthquake.

The scale generally goes from 1 to 9 (though it doesn't really have an upper limit), but an earthquake which has an intensity of 6 on the Richter scale is 10 times more powerful than an earthquake which measures 5.

One of the strongest earthquakes in history occurred in Chile on May 22, 1960. This earthquake measured an amazing 9.5 on the Richter scale—over 30,000 times stronger than a magnitude 5 earthquake!

PROBLEM 1 Exponential Growth



A famous legend tells the story of the inventor of the game of chess. When the inventor showed the new game to the emperor of India, the emperor was so astonished, he said to the inventor, "Name your reward!"

The wise inventor asked the emperor for 1 grain of rice for the first square of the chessboard, 2 grains for the second square, 4 grains for the third square, 8 grains for the fourth square, and so on.



1. Determine the number of grains of rice on the fifth, sixth, seventh, and eighth squares. Write your answers in the second column of the table.

Square Number	Number of Rice Grains	Power
1	1	
2	2	
3	4	
4	8	
5		
6		
7		
8		
s		

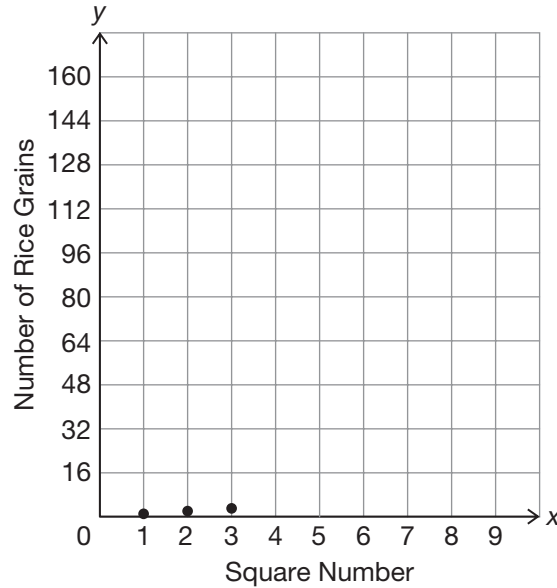
2. What pattern do you notice in the table?



3. A chessboard has 64 squares. Predict how many grains of rice must be on the 64th square if the pattern continues.



4. Graph the points from your table. The first few points have been plotted.



5. Does it make sense to connect the points in this graph? Explain why or why not. If so, connect the points.

6. Complete the third column of the table in Question 1. Write the appropriate power to represent the number of rice grains on the first 8 squares of the chessboard.

7. Analyze your table and write an algebraic expression in the last row to determine the number of rice grains for any square number.

- a. Write an equation in function notation to represent the number of rice grains as a function of the square number, s .

- b. Explain how you determined your equation.



8. Pat and George each wrote a function to represent the number of rice grains for any square number using different methods.

 **Pat**

I compared the exponents of the power to the square number in the table. Each exponent is 1 less than the square number.

$$f(s) = 2^{s-1}$$

 **George**

I know this is an exponential function with a common base of 2. If I extend the pattern back on the graph I get the y-intercept of

$$\left(0, \frac{1}{2}\right), \text{ so } a = \frac{1}{2}.$$

$$f(s) = \frac{1}{2}(2)^s$$

Use properties of exponents to verify that 2^{s-1} and $\frac{1}{2}(2)^s$ are equivalent.



9. Use the **intersection** feature of a graphing calculator to answer each question. Write each answer as an equation or compound inequality. Explain how you determined your answer.
- a. Which square on the chessboard contains 262,144 rice grains?

Make sure you adjust the settings for your graph window so that you can answer each question!

- b. Which square on the chessboard contains 32,768 rice grains?



c. Which chess squares on the chessboard contain fewer than 20,000 rice grains?

d. Which squares on the chessboard contain at least 1 million rice grains?

10. Use your calculator to determine the number of rice grains that would be on the very last square of the chessboard.



11. In Question 3, you predicted the number of grains of rice on the last square of the chessboard. How close was your prediction to the actual answer?

PROBLEM 2 Using Properties of Exponents



An exponential function is a function written in the form $f(x) = a \cdot b^x$. To solve an exponential function, you can use what you know about the properties of exponents and common bases.



To solve the exponential equation $\frac{1}{32} = 2^{x-1}$, first determine the power of 2 that gives the result of 32:

$$(2)(2)(2)(2)(2) = 32$$

$$2^5 = 32$$

Remember what you've learned about negative exponents.



Then rewrite the equation to show common bases:

$$\frac{1}{32} = 2^{x-1}$$

$$\frac{1}{2^5} = 2^{x-1}$$

$$2^{-5} = 2^{x-1}$$



Because the expressions on both sides of the equals sign have the same base, you can set up and solve an equation using the exponents:

$$-5 = x - 1$$

$$-5 + 1 = x - 1 + 1$$

$$-4 = x$$

$$\text{Therefore, } x = -4.$$



5



Solve each equation for x .

1. $3^x = 81$

2. $2^{4x} = 1$

3. $4^{8-x} = \frac{1}{64}$



4. $5^{9x} = 5$

5. $\frac{1}{3^{x+5}} = 243$

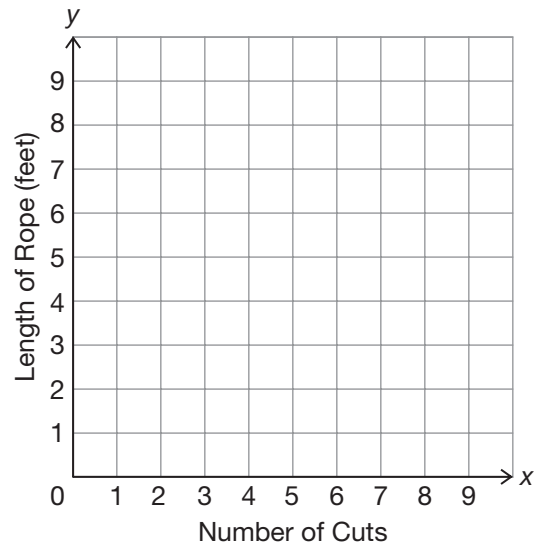
6. $2^{-x} = \frac{1}{2}$

PROBLEM 3 Inverses



1. The Amazing Aloysius is practicing one of his tricks. As part of the trick, he cuts a rope into many pieces and then magically puts the pieces of rope back together. He begins the trick with a 10-foot rope and then cuts it in half. He takes one of the halves and cuts that piece in half. He keeps cutting the pieces in half until he is left with a piece so small he can't cut it anymore.
- a. Complete the table to show the length of rope after each of Aloysius's cuts. Write each length as a whole number, mixed number, or fraction. Then graph the points from the table.

Number of Cuts	Length of Rope (feet)
0	10
1	
2	
3	
4	
c	



- b. Determine which function family represents this situation. Then, write the length of the rope as a function of the cut number.

c. Use your function to determine the length of the rope after the 7th cut.

2. Write an exponential function of the form $f(x) = ab^x$ given each pair of ordered pairs.

a. $(0, 4)$ and $(3, \frac{1}{2})$

b. $(-2, -\frac{1}{8})$ and $(0, -2)$



Be prepared to share your solutions and methods.

Chapter 5 Summary

KEY TERMS

- simple interest (5.1)
- compound interest (5.1)
- horizontal asymptote (5.2)
- basic function (5.3)
- transformation (5.3)
- vertical translation (5.3)
- coordinate notation (5.3)
- argument of a function (5.3)
- horizontal translation (5.3)
- reflection (5.4)
- line of reflection (5.4)
- cube root (5.5)
- index (5.5)
- n th root (5.5)
- radicand (5.5)
- rational exponent (5.5)

5.1 Writing and Solving Simple Interest Equations

In a simple interest account, the interest you earn at the end of each year is a percent of the original principal. The pattern of the account balance year after year creates an arithmetic sequence because the same amount is added to each term to create the next term in the sequence. The sequence can be written as a function in terms of time: $P(t) = P_0 + (P_0 \cdot r)t$, where t is time, $P(t)$ is the total balance after t years, P_0 is the original principal, and r is the interest rate written as a decimal.

Example

An account that starts with \$500 and earns 3% in simple interest each year will reach \$725 in 15 years.

$$\begin{aligned}P(t) &= P_0 + (P_0 \cdot r)t \\725 &= 500 + (500 \cdot 0.03)t \\225 &= 15t \\15 &= t\end{aligned}$$

5.1 Writing and Solving Compound Interest Equations

In a compound interest account, the interest you earn at the end of each year is a percent of the account balance at the beginning of the year. The pattern of the account balance year after year creates a geometric sequence because the same amount is multiplied by each term in the sequence to produce the next. The sequence can be written as a function in terms of time: $P(t) = P_0 \cdot (1 + r)^t$, where t is time, $P(t)$ is the total balance after t years, P_0 is the initial deposit amount, and r is the interest rate each year, written as a decimal.

Example

After 10 years, an account that starts with \$500 and earns 3% compound interest each year will have \$671.96.

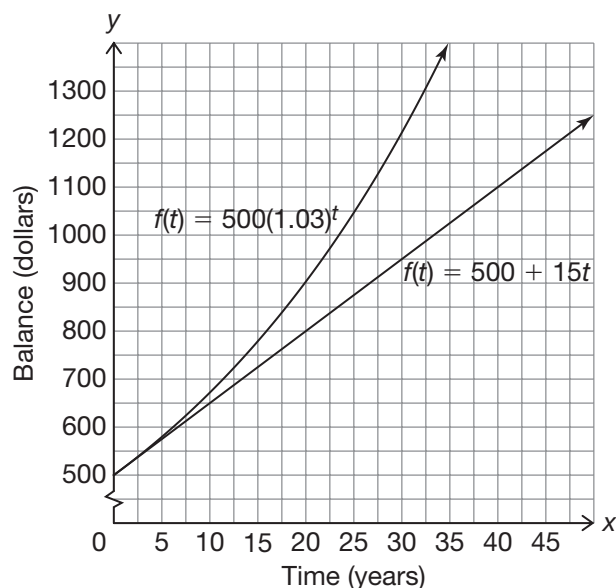
$$\begin{aligned}P(t) &= P_0 \cdot (1 + r)^t \\&= 500 \cdot (1 + 0.03)^{10} \\&\approx 500 \cdot 1.34392 \\&\approx 671.96\end{aligned}$$

5.1 Comparing Simple and Compound Interest

The rate of change for a simple interest account is constant. The rate of change between the values for the compound interest account is increasing as t becomes larger. A constant rate of change means that the graph of the linear equation is a straight line. An increasing rate of change means that the graph of the exponential function is a smooth curve.

Example

Quantity	Time	Simple Interest Balance	Compound Interest Balance
Units	years	dollars	dollars
Expression	t	$500 + 15t$	$500(1.03)^t$
	0	500	500
	1	515	515
	2	530	530.45
	10	650	671.96
	100	2000	9609.32



5.2 Writing and Solving Equations for Population Problems

The amount of population growth over time is not constant, so it can be compared to the compound interest formula: $P(t) = P(1 + r)^t$, where the starting population is represented by P , the rate of population growth is r , and time is t . When the common ratio, $1 + r$, is greater than 1, the exponential function increases. When the common ratio is between 0 and 1, the exponential function decreases.

Example

A population that starts at 50,000 grows at a rate of 2% every year. After 10 years, the population will be 60,950.

$$\begin{aligned}
 P(t) &= P(1 + r)^t \\
 &= 50,000(1 + 0.02)^{10} \\
 &\approx 50,000(1.219) \\
 &\approx 60,950
 \end{aligned}$$

5.2 Solving Population Problems for Time Using Intersecting Points

If $P(t)$ is the function for a population's growth, then you can use the **intersection** feature on a graphing calculator to determine the time it will take to reach a given population.

Examples

A population starts at 5000 and grows at a rate of 1.7% every year. To determine when the population will double, use a graphing calculator to graph each function.

$$P(t) = 5000(1 + 0.017)^t$$

$$P(t) = 10,000$$

The graphs intersect between $x = 41$ and $x = 42$. So, the population will double in a little over 41 years.

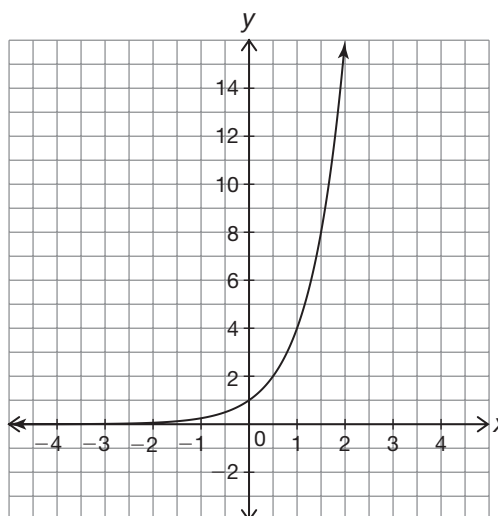
5.2 Graphing and Analyzing Exponential Functions

A horizontal asymptote is a horizontal line that a function gets closer and closer to, but never intersects.

Example

$$f(x) = 4^x$$

x	$f(x)$
-2	$\frac{1}{16}$
-1	$\frac{1}{4}$
0	1
1	4
2	16
3	64



x -intercept(s): none

y -intercept: $(0, 1)$

asymptote: $y = 0$

domain: all real numbers

range: $y > 0$

interval(s) of increase/decrease: increasing over the entire domain

5.3

Translating Linear and Exponential Functions in Terms of the Basic Function

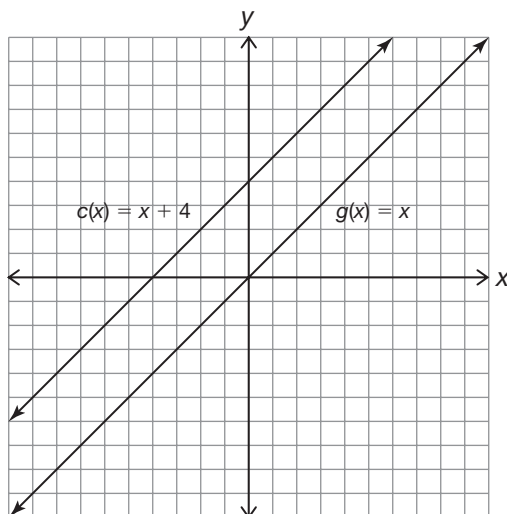
A basic function is the simplest function of its type. For example, $f(x) = x$ is the simplest linear function. It is in the form $f(x) = ax + b$, where $a = 1$ and $b = 0$. You can write a vertically translated function in terms of the basic function $f(x)$. A transformation is the mapping, or movement, of all the points of a figure in a plane according to a common operation. A vertical translation is a type of transformation that shifts the entire graph up or down.

Examples

a. $g(x) = x$ and $c(x) = x + 4$

$$c(x) = g(x) + 4$$

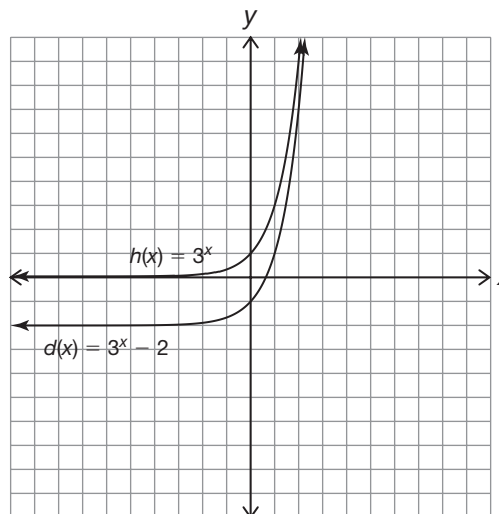
For every x -coordinate on the graph of $g(x)$, the y -coordinate of the graph of $c(x)$ will be 4 more than the y -coordinate of the graph of $g(x)$.



b. $h(x) = 3^x$ and $d(x) = 3^x - 2$

$$d(x) = h(x) - 2$$

For every x -coordinate on the graph of $h(x)$, the y -coordinate of the graph of $d(x)$ will be 2 less than the y -coordinate of the graph of $h(x)$.



5

5.3

Translating Linear and Exponential Functions Vertically Using Coordinate Notation

A vertical translation affects the y -coordinate of each point on the graph. You can use the coordinate notation shown to indicate a vertical translation:

$$(x, y) \rightarrow (x, y + b), \text{ where } b \text{ is a real number.}$$

Examples

a. $g(x) = x$ and $f(x) = x - 3$

$$(x, y) \rightarrow (x, y - 3)$$

b. $j(x) = 4^x$ and $k(x) = (4^x) + 5$

$$(x, y) \rightarrow (x, y + 5)$$

5.3 Translating Linear and Exponential Functions Vertically

A vertical translation is a type of transformation that shifts the entire graph up or down. Suppose $g(x) = x$ is the basic function and $f(x) = x + b$. When $b > 0$, the graph of $f(x)$ is b units above the graph of $g(x)$. When $b < 0$, the graph of $f(x)$ is b units below the graph of $g(x)$. Suppose $h(x) = b^x$ is the basic function and $f(x) = b^x + k$. When $k > 0$, the graph of $f(x)$ is k units above the graph of $h(x)$. When $k < 0$, the graph of $f(x)$ is k units below the graph of $h(x)$.

Examples

- The equation of the function $p(x) = 2x$ after a vertical translation 3 units up is:
 $q(x) = 2x + 3$.
- The equation of the function $m(x) = 6^x$ after a vertical translation 3 units down is:
 $r(x) = 6^x - 3$.

5.3 Translating Linear and Exponential Functions Horizontally in Terms of the Basic Function

The operations that produce horizontal translations are done on x -values, which is the argument of the function. The argument of a function is the variable on which the function operates. To write $v(x)$ in terms of the basic function $h(x)$, substitute the argument for $v(x)$ into the argument for $h(x)$.

Examples

- $b(x) = x$ and $c(x) = (x + 3)$

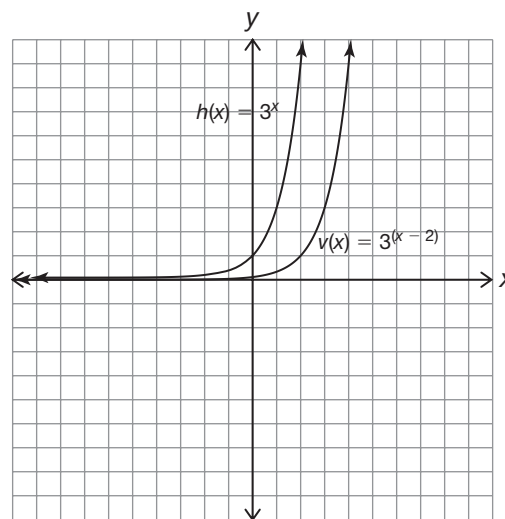
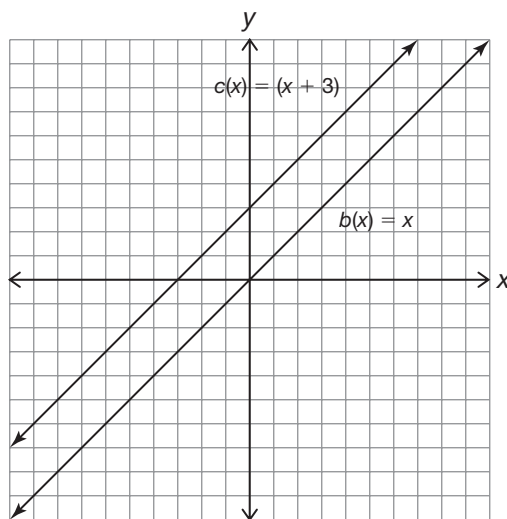
$$b(x + 3) = (x + 3) = c(x)$$

For the same y -coordinate, the x -coordinate of $c(x)$ is 3 less than the x -coordinate of $b(x)$.

- $h(x) = 3^x$ and $v(x) = 3^{(x-2)}$

$$h(x - 2) = 3^{(x-2)} = v(x)$$

For the same y -coordinate, the x -coordinate of $v(x)$ is 2 more than the x -coordinate of $h(x)$.



5.3

Translating Linear and Exponential Functions Horizontally Using Coordinate Notation

A horizontal translation of a graph is a shift of the entire graph left or right. A horizontal translation affects the x -coordinate of each point on the graph. You can use the coordinate notation shown to indicate a horizontal translation:

$$(x, y) \rightarrow (x + a, y), \text{ where } a \text{ is a real number.}$$

Examples

a. $g(x) = x$ and $f(x) = (x - 7)$

$$(x, y) \rightarrow (x + 7, y)$$

b. $j(x) = 6^x$ and $k(x) = 6^{(x+4)}$

$$(x, y) \rightarrow (x - 4, y)$$

5.3

Translating Linear and Exponential Functions Horizontally

A horizontal translation of a graph is a shift of the entire graph left or right. Suppose $h(x) = x$ is the basic function and $f(x) = (x - c)$. When $c > 0$, then the graph of $f(x)$ is c units to the right of the graph of $h(x)$. When $c < 0$, the graph of $f(x)$ is c units to the left of the graph of $h(x)$. Suppose $h(x) = b^x$ is the basic function and $f(x) = b^{x-c}$. When $c > 0$, the graph of $f(x)$ is c units to the right of the graph of $h(x)$. When $c < 0$, the graph of $f(x)$ is c units to the left of the graph of $h(x)$.

Examples

a. The equation of the function $h(x) = 2x^3$ after a horizontal translation 5 units right is

$$j(x) = 2(x - 5)^3.$$

b. The equation of the function $c(x) = 6^x$ after a horizontal translation 5 units left is

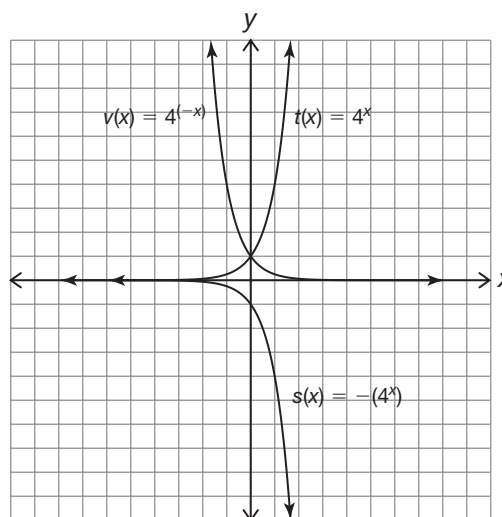
$$k(x) = 6^{(x+5)}.$$

5.4 Reflecting Linear and Exponential Functions Vertically and Horizontally in Terms of the Basic Function

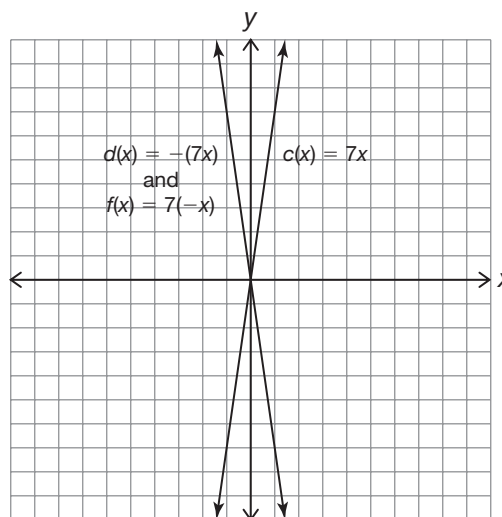
A reflection of a graph is a mirror image of the graph about a line. The line that the graph is reflected over is called the line of reflection. A basic function is multiplied by -1 to result in a reflection over the x -axis, or $y = 0$. The argument of a basic function is multiplied by -1 to result in a reflection over the y -axis, or $x = 0$.

Examples

- a. $t(x) = 4^x$ is the basic function.
 $s(x) = -(4^x)$ is a reflection of the function $t(x)$ over $y = 0$.
 $v(x) = 4^{(-x)}$ is a reflection of the function $t(x)$ over $x = 0$.



- b. $c(x) = 7x$ is the basic function.
 $d(x) = -(7x)$ is a reflection of the function $c(x)$ over $y = 0$.
 $f(x) = 7(-x)$ is a reflection of the function $c(x)$ over $x = 0$.



5.4 Reflecting Exponential Functions Vertically and Horizontally Using Coordinate Notation

A reflection of a graph is a mirror image of the graph about a line. The line that the graph is reflected about is called the line of reflection. A horizontal line of reflection affects the y -coordinates. Use the coordinate notation $(x, y) \rightarrow (x, -y)$ to indicate a reflection about a horizontal line of reflection. A vertical line of reflection affects the x -coordinates. Use the coordinate notation $(x, y) \rightarrow (-x, y)$ to indicate a reflection about a vertical line of reflection.

Examples

a. $g(x) = 3x$ and $f(x) = -(3x)$
horizontal line of reflection
 $(x, y) \rightarrow (x, -y)$

b. $j(x) = 3^x$ and $k(x) = 3^{(-x)}$
vertical line of reflection
 $(x, y) \rightarrow (-x, y)$

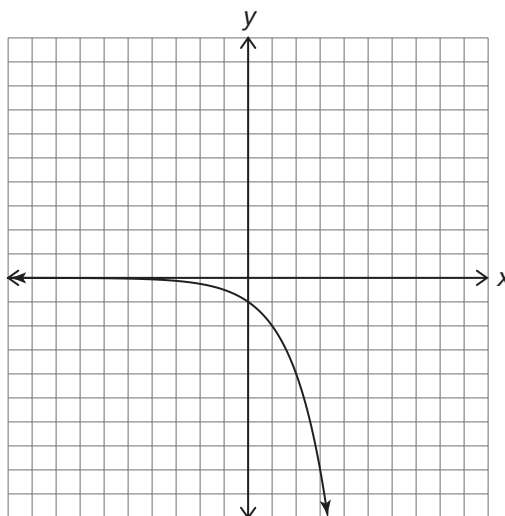
5.4 Determining Characteristics of Graphs after Transformations

A graph and its function can be drawn or identified from its description as linear or exponential, increasing or decreasing, and continuous or discrete. A graph or function can also be described as a transformation of a basic function: a vertical translation, a horizontal translation, a reflection over the line $y = 0$, or a reflection over the line $x = 0$.

Examples

The graph and equation represent an exponential function that is continuous, is decreasing, and is a reflection of $f(x) = 2^x$ over the line $y = 0$.

$$g(x) = -(2^x)$$



5.5 Simplifying Expressions with Negative Exponents

To write a negative power as a positive power, write the power in the denominator of a fraction whose numerator is 1 and change the sign of the exponent in the power from negative to positive.

Examples

$$20^{-2} = \frac{20^0}{20^2} = \frac{1}{400}$$

5.5 Evaluating n th Roots in Radical Form

You can say that a number b is a cube root of a number a if $b^3 = a$. A cube root is indicated by the symbol $\sqrt[3]{}$. The number 3 is called the index of the radical. Extending the idea of square and cube roots, use n to represent a positive number. Then, a number b is the n th root of a if $b^n = a$. The n th root of a number a is designated as $\sqrt[n]{a}$, where n is the index of the radical and a is the radicand.

Examples

a. $\sqrt[3]{-343} = -7$ because $(-7)^3 = -343$

b. $\sqrt[6]{729} = 3$ because $3^6 = 729$

5.5 Writing Radicals as Powers and Powers as Radicals

A rational exponent is an exponent that is a rational number. You can write each n th root using a rational exponent. If n is an integer greater than 1, then $\sqrt[n]{a} = a^{\frac{1}{n}}$.

Examples

a. $\sqrt[4]{b} = b^{\frac{1}{4}}$

b. $6^{\frac{1}{5}} = \sqrt[5]{6}$

5.5 Writing Expressions in Radical Form and Rational Exponent Form

Write expressions with rational exponents in radical form by using the properties you know about powers. Write the power as a product using a unit fraction. Use the power of a power rule and the definition of a rational exponent to write the power as a radical.

Examples

a. $8^{\frac{2}{3}} = 8^{\left(\frac{1}{3}\right)(2)}$
 $= (\sqrt[3]{8})^2$

b. $(\sqrt[7]{c})^3 = c^{\left(\frac{1}{7}\right)(3)}$
 $= c^{\frac{3}{7}}$

5.6 Writing Exponential Equations

Use a table of values to determine an exponential pattern. Define an independent variable and write an algebraic expression for the dependent variable. Then, write an equation in function notation to represent the change as a function of the independent variable.

Example

Gavin starts with one plant and each year finds that more plants have grown.

Time (years)	Number of Plants	Expression
1	1	3^0
2	3	3^1
3	9	3^2
4	27	3^3

The function $f(t) = 3^{t-1}$ is the number of plants Gavin has in t years.

5.6 Graphing to Solve Exponential Equations

Use the **intersection** feature on a graphing calculator to determine a specific independent variable value given a dependent value. Determine where the graph of $f(x)$ will intersect with a line representing a given $f(x)$ to determine the value of x .

Example

Gavin's plants are multiplying according to the function $f(t) = 3^{t-1}$. He wants to determine when he will have 6561 plants.

The intersection point of the graphs of $f(t) = 3^{t-1}$ and $f(t) = 6561$ is at (9, 6561). Gavin will have 6561 plants in 9 years.

5.6 Solving Exponential Equations

An exponential function is a function written in the form $f(x) = a \cdot b^x$. To solve an exponential function, you can use what you know about the properties of exponents. To solve an exponential equation, first determine the power of the base that gives the result. Rewrite the equation in terms of the exponent. Then determine what x must be to equal that exponent.

Example

$$5^{x-3} = 625$$

$$5 \cdot 5 \cdot 5 \cdot 5 = 5^4 = 625$$

$$5^{x-3} = 5^4$$

$$\text{So, } x - 3 = 4.$$

$$\text{Therefore, } x = 7.$$