Geometry on the Coordinate Plane

12

12.1

Let's Move!

Translating and Constructing Line Segments

LEARNING GOALS

In this lesson, you will:

- Determine the distance between two points.
- Use the Pythagorean Theorem to derive the Distance Formula.
- Apply the Distance Formula on the coordinate plane.
- Translate a line segment on the coordinate plane.
- Copy or duplicate a line segment by construction.

KEY TERMS

- Distance Formula
- transformation
- rigid motion
- translation
- image
- pre-image
- arc
- congruent line segments
- congruent

CONSTRUCTIONS

- copying a line segment
- duplicating a line segment

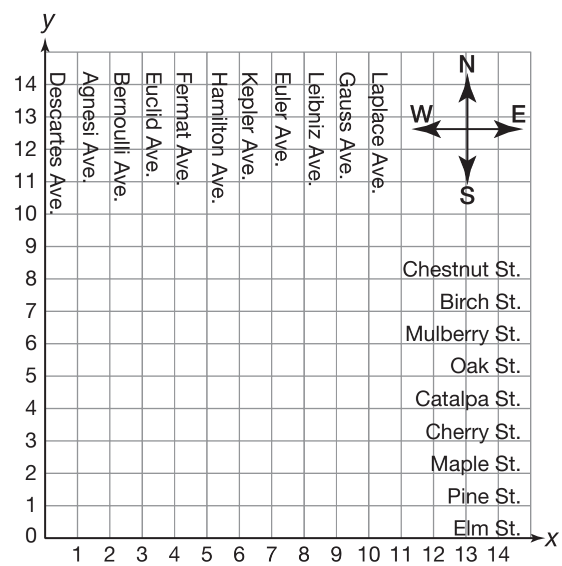
Are you better at geometry or algebra? Many students have a preference for one subject or the other. However, geometry and algebra are very closely related. While there are some branches of geometry that do not use much algebra, analytic geometry applies methods of algebra to geometric questions. Analytic geometry is also known as the study of geometry using a coordinate system. So anytime you are studying geometry and it involves a coordinate system, you are working on analytic geometry. Be sure to thank Descartes and his discovery of the coordinate plane for this!

What might be the pros and cons of analytic geometry compared to other branches of geometry? Does knowing about analytic geometry change how you feel about your own abilities in geometry or algebra?

PROBLEM 1 Where Are You?

Don, Freda, and Bert live in a town where the streets are laid out in a grid system.

1. Don lives 3 blocks east of Descartes Avenue and 5 blocks north of Elm Street. Freda lives 7 blocks east of Descartes Avenue and 2 blocks north of Elm Street. Plot points to show the locations of Don's house and Freda's house on the coordinate plane. Label each location with the student's name and the coordinates of the point.



a. Name the streets that Don lives on.

b. Name the streets that Freda lives on.

12

2. Bert lives at the intersection of the avenue that Don lives on and the street that Freda lives on. Plot and label the location of Bert's house on the coordinate plane. Describe the location of Bert's house with respect to Descartes Avenue and Elm Street.

3. How do the x- and y-coordinates of Bert's house compare to the x- and y-coordinates of Don's house and Freda's house?

4. Use Don's and Bert's house coordinates to write and simplify an expression that represents the distance between their houses. Explain what this means in terms of the problem situation.

5. Use Bert's and Freda's house coordinates to write and simplify an expression that represents the distance between their houses. Explain what this means in terms of the problem situation.

6. All three friends are planning to meet at Don's house to hang out. Freda walks to Bert's house, and then Freda and Bert walk together to Don's house.

   a. Use the coordinates to write and simplify an expression that represents the total distance from Freda's house to Bert's house to Don's house.

   b. How far, in blocks, does Freda walk altogether?

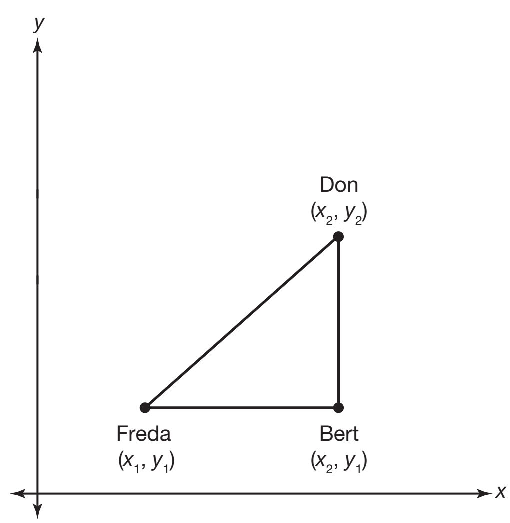
12

7. Draw the direct path from Don's house to Freda's house on the coordinate plane. If Freda walks to Don's house on this path, how far, in blocks, does she walk? Explain how you determined your answer.

8. Complete the summary of the steps that you took to determine the direct distance between Freda's house and Don's house. Let d be the direct distance between Don's house and Freda's house.

Distance between Bert's house and Freda's house		Distance between Don's house and Bert's house		Direct distance between Don's house and Freda's house
$(\square - \square)^2$	+	$(\square - \square)^2$		$= \square$
$\square^2$	+	$\square^2$		$= \square$
$\square$	+	$\square$		$= \square$
				$\square = \square$
				$\square = \square$

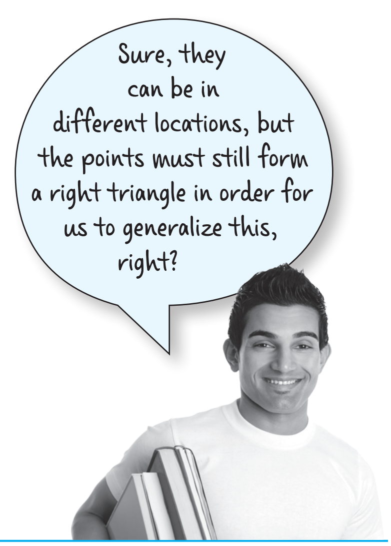
Suppose Freda's, Bert's, and Don's houses were at different locations. You can generalize their locations by using x_1, x_2, y_1, and y_2 and still solve for the distances between their houses.



9. Use the graph to determine the distance from:

a. Don's house to Bert's house (DB).

b. Bert's house to Freda's house (BF).

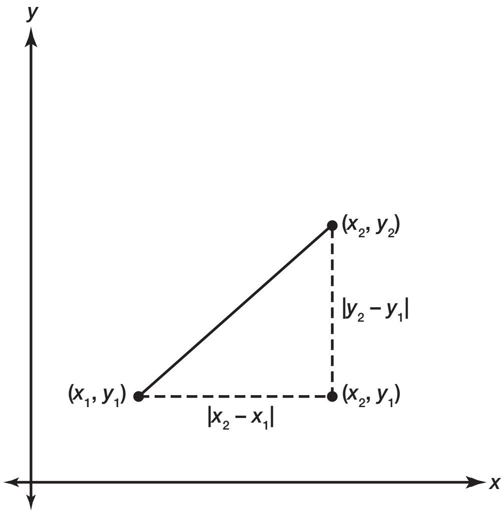


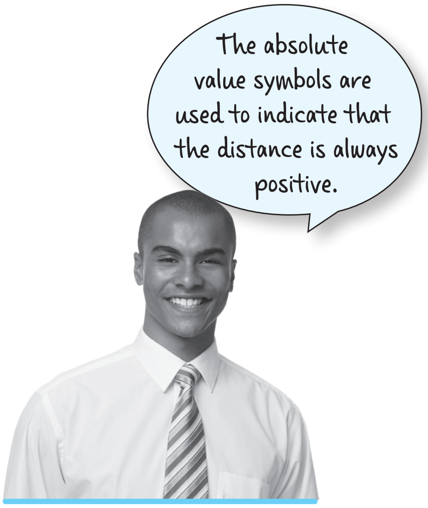
10. Use the Pythagorean Theorem to determine the distance from Don's house to Freda's house (DF).

You used the Pythagorean Theorem to calculate the distance between two points on the coordinate plane. Your method can be written as the *Distance Formula.*

The **Distance Formula** states that if (x_1, y_1) and (x_2, y_2) are two points on the coordinate plane, then the distance d between (x_1, y_1) and (x_2, y_2) is given by

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.





11. Do you think that it matters which point you identify as (x_1, y_1) and which point you identify as (x_2, y_2) when you use the Distance Formula? Explain your reasoning.

12

12. Calculate the distance between each pair of points. Round your answer to the nearest tenth if necessary. Show all your work.

a. (1, 2) and (3, 7)

b. (−6, 4) and (2, −8)

c. (−5, 2) and (−6, 10)

13. Carlos and Mandy just completed Question 12 parts (a) through (c). They need to calculate the distance between the points (−1, −2) and (−3, −7). They notice the similarity between this problem and part (a). Carlos says that the solution must be the negative of the solution of part (a) of Question 12. Mandy disagrees and says that the solution will be the same as the solution of part (a). Who is correct? Explain your reasoning and state the correct solution.

12

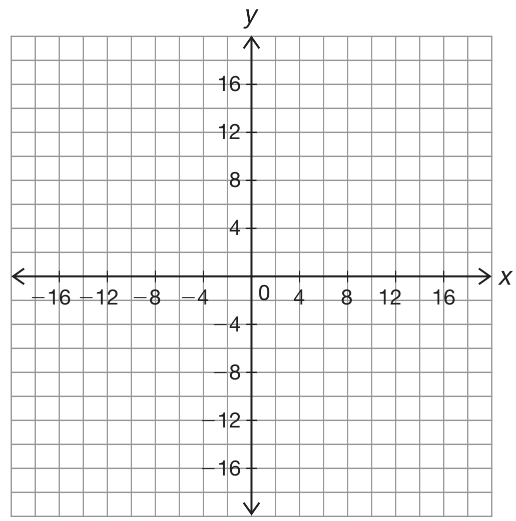
14. The distance between (x, 2) and (0, 6) is 5 units. Use the Distance Formula to determine the value of x. Show all your work.

PROBLEM 2 Translating a Line Segment

1. Pedro's house is located at (6, 10). Graph this location on the coordinate plane and label the point *P*.

2. Jethro's house is located at (2, 3). Graph this location on the coordinate plane and label the point *J*.

3. Draw a line connecting the two houses to create line segment *PJ*.



4. Determine the length of line segment *PJ*.

A **transformation** is the mapping, or movement, of all the points of a figure in a plane according to a common operation.

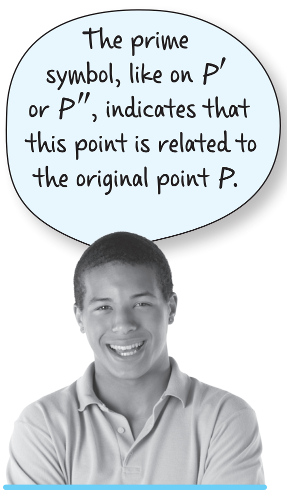
A **rigid motion** is a transformation of points in space.

A **translation** is a rigid motion that "slides" each point of a figure the same distance and direction. Sliding a figure left or right is a horizontal translation, and sliding it up or down is a vertical translation. The new figure created from the translation is called the **image**. The original figure is called the **pre-image**.

5. Line segment PJ is horizontally translated 10 units to the left.

a. Graph the new location of line segment PJ. Label the new points P' and J'.

b. Identify the coordinates of P' and J'.



6. Line segment $P'J'$ is vertically translated 14 units down.

a. Graph the new location of line segment $P'J'$. Label the new points P'' and J''.

b. Identify the coordinates of P'' and J''.

12

7. Line segment $P''J''$ is horizontally translated 10 units to the right.

a. Without graphing, predict the coordinates of P''' and J'''.

b. Graph the new location of line segment $P''J''$. Label the new points P''' and J'''.

8. Describe the translation necessary on line segment $P'''J'''$ so that it returns to the location of PJ.

9. How do the lengths of the images compare to the lengths of the pre-images? Explain how you could verify your answer.

10. Analyze the coordinates of the endpoints of each line segment.

   a. Identify the coordinates of each line segment in the table.

Line Segments	PJ	$P'J'$	$P''J''$	$P'''J'''$
Coordinates of Endpoints				

   b. Describe how a horizontal translation changes the coordinates of the endpoints.

   c. Describe how a vertical translation changes the coordinates of the endpoints.

11. How many translations occurred for the line segment to return to its original location?

12

12. Does either a vertical or horizontal translation of a line segment alter the length of the line segment? Explain why or why not.

13. If both points on a line segment were not moved the same distance or direction, would the length of the line segment change? Would this still be considered a translation? Explain your reasoning.

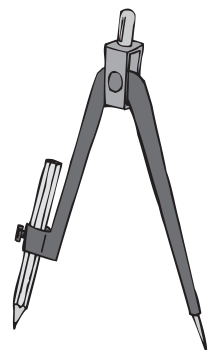
PROBLEM 3 Copying Line Segments

In the previous problem, you translated line segments on the coordinate plane. The lengths of the line segments on the coordinate plane are measurable.

In this problem, you will translate line segments when measurement is not possible. This basic geometric construction is called **copying a line segment** or **duplicating a line segment.** The construction is performed using a compass and a straightedge.

One method for copying a line segment is to use circles. But before you can get to that, let's review how to draw perfect circles with a compass.

Remember that a compass is an instrument used to draw circles and arcs. A compass can have two legs connected at one end.



One leg has a point, and the other holds a pencil. Some newer compasses may be different, but all of them are made to construct circles by placing a point firmly into the paper and then spinning the top of the compass around, with the pencil point just touching the paper.

1. Use your compass to construct a number of circles of different sizes.

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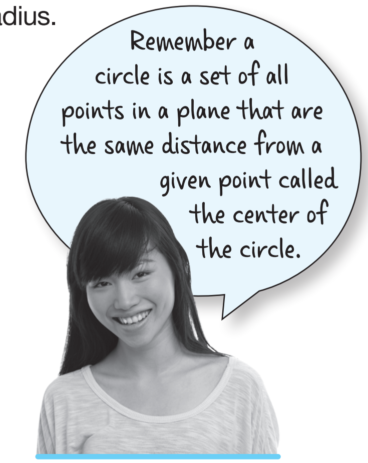
2. Point C is the center of a circle and line segment CD is the radius.

   a. Construct circle C.



   b. Draw and label points A, B, E, and F anywhere on the circle.

   c. Construct $\overline{AC}$, $\overline{BC}$, $\overline{EC}$, and $\overline{FC}$.

   d. What conclusion can you make about all these line segments? Explain your reasoning.

   e. Do you think the line segments you constructed are also radii of the circle? How do you know?

An **arc** is a part of a circle. You can also think of an arc as the curve between two points on the circle.

3. Point C is the center of a circle and line segment AC is the radius.

   a. Construct an arc of circle C. Make your arc about one-half inch long. Construct the arc so that it does not pass through A.

   b. Draw and label two points B and E, on the arc and construct line segments CE and CB.

   c. What conclusion can you make about these line segments?

Line segments that have the same length are called **congruent line segments**. **Congruent** means to have the same size, shape, and measure. You can indicate that two line segments are congruent by using the congruence symbol, $\cong$, and writing the names of the line segments that are congruent on either side of it. For example, $\overline{CB} \cong \overline{CA}$ is read as "line segment *CB* is congruent to line segment *CA*."

4. Construct a circle with the center *A* and a radius of about 1 inch.
   a. Without changing the width of your compass, place the compass point on any point on the circle you constructed and then construct another circle.
   b. Draw a dot on a point where the two circles intersect. Place the compass point on that point of intersection of the two circles, and then construct another circle.
   c. Repeat this process until no new circles can be constructed.
   d. Connect the points of the circles' intersections with each other.

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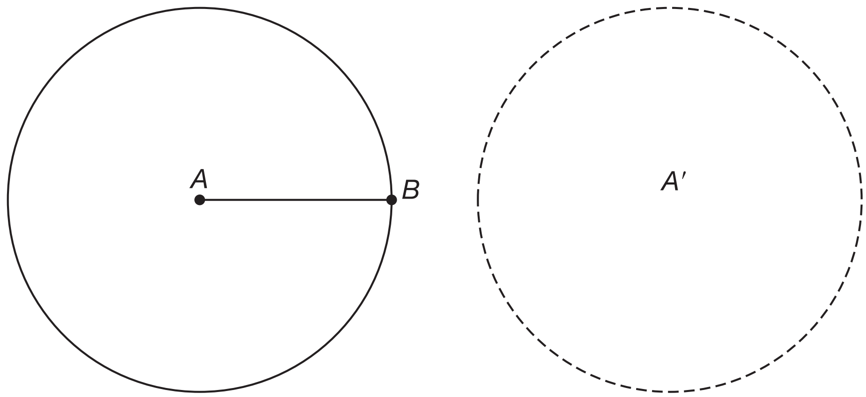
12

   e. Describe the figure formed by the line segments.

Now let's use these circle-drawing skills to duplicate a line segment.

5. Circle A is congruent to Circle A'.



a. Duplicate line segment AB in Circle A'. Use point A' as the center of the circle, then label the endpoint of the duplicated segment as point B'.

b. Describe the location of point B'.

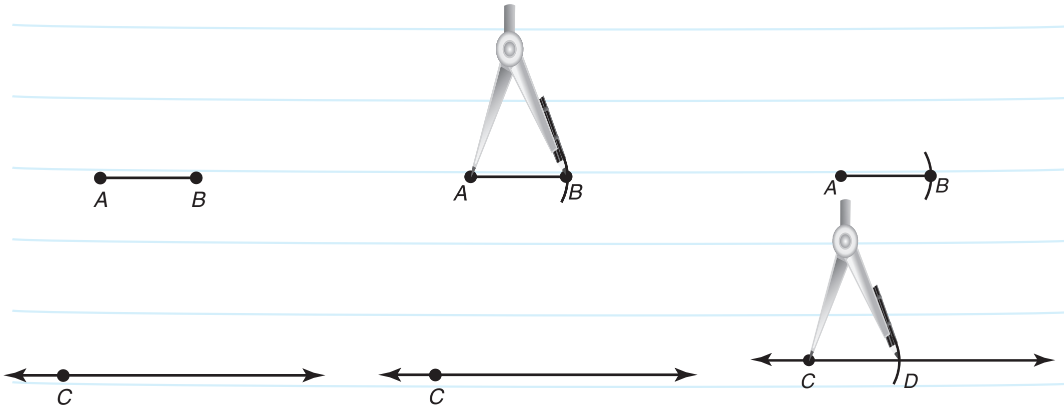
c. If possible, draw a second line segment in Circle A' that is a duplicate of line segment AB. Label the duplicate segment line segment $A'C'$. If it is not possible, explain why.

12

PROBLEM 4 Another Method

Line segments can also be duplicated while using a compass, but not drawing a circle.

You can duplicate a line segment by constructing an exact copy of the original line segment.



Construct a Starter Line

Use a straightedge to construct a starter line longer than segment *AB*. Label point *C* on the line.

Measure Length

Set your compass at the length *AB*.

Copy Length

Place the compass at *C*. Mark point *D* on the new segment.

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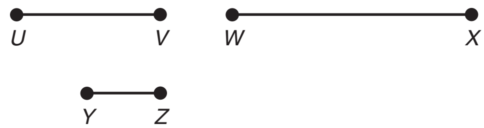
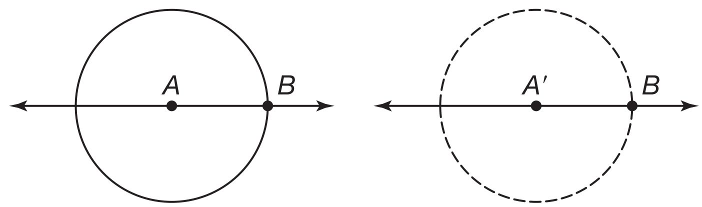
1. Construct a line segment that is twice the length of line segment *AB*.

2. Duplicate each line segment using a compass and a straightedge.

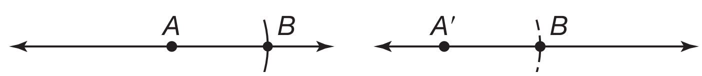


3. Dave and Sandy are duplicating the segment *AB*. Their methods are shown.

Dave



Sandy



Which method is correct? Explain your reasoning.

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4. Which method do you prefer? Why?

Talk the Talk

Translating a line segment on the coordinate plane using coordinates and translating a line segment by construction using construction tools both preserve distance.

1. How are the two methods of translation similar?

2. How are the two methods of translation different?

12

Be prepared to share your solutions and methods.

12.2

Treasure Hunt

Midpoints and Bisectors

LEARNING GOALS

In this lesson, you will:

- Determine the midpoint of a line segment on a coordinate plane.
- Use the Midpoint Formula.
- Apply the Midpoint Formula on the coordinate plane.
- Bisect a line segment using patty paper.
- Bisect a line segment by construction.
- Locate the midpoint of a line segment.

KEY TERMS

- midpoint
- Midpoint Formula
- segment bisector

CONSTRUCTIONS

- bisecting a line segment

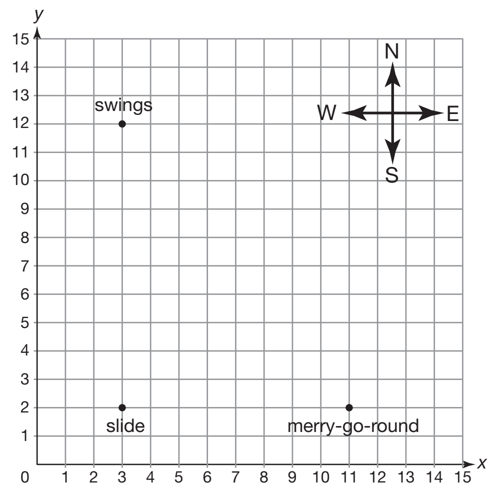
When you hear the phrase "treasure hunt," you may think of pirates, buried treasure, and treasure maps. However, there are very few documented cases of pirates actually burying treasure, and there are no historical pirate treasure maps! So where did this idea come from?

Robert Louis Stevenson's book *Treasure Island* is a story all about pirates and their buried gold, and this book greatly influenced public knowledge of pirates. In fact, it is Stevenson who is often credited with coming up with the concept of the treasure map and using an X to mark where a treasure is located.

Have you ever used a map to determine your location or the location of another object? Did you find it difficult or easy? How does the idea of a treasure map relate to a mathematical concept you are very familiar with?

PROBLEM 1 Locating the Treasure

Ms. Lopez, a kindergarten teacher, is planning a treasure hunt for her students. She drew a model of the playground on a coordinate plane as shown. She used this model to decide where to place items for the treasure hunt, and to determine how to write the treasure hunt instructions. Each grid square represents one square yard on the playground.



1. Determine the distance, in yards, between the merry-go-round and the slide. Show all your work.

2. Ms. Lopez wants to place a small pile of beads in the grass halfway between the merry-go-round and the slide. How far, in yards, should the beads be placed from the merry-go-round and the slide? Write the coordinates for the location exactly halfway between the merry-go-round and the slide.

12

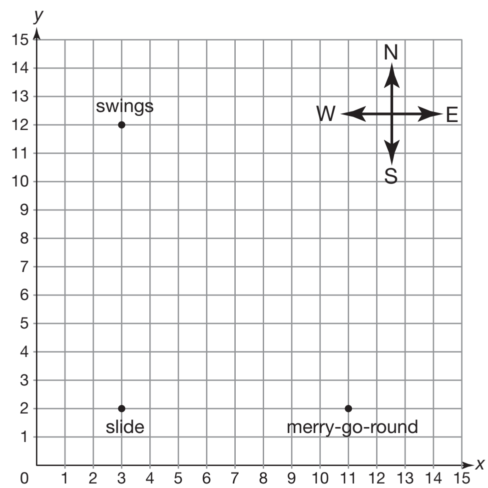
3. Graph the location of the beads on the coordinate plane.

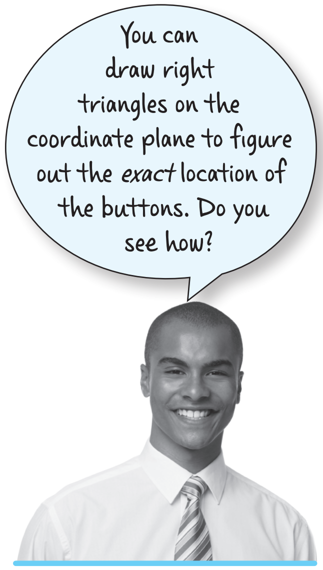
4. How do the coordinates of the location of the beads compare to the coordinates of the locations of the slide and merry-go-round?

5. Ms. Lopez wants to place a pile of kazoos in the grass halfway between the slide and the swings. What should the coordinates of the location of the kazoos be? Explain your reasoning. Plot and label the location of the kazoos on the coordinate plane.

6. How do the coordinates of the location of the kazoos compare to the coordinates of the locations of the slide and swings?

7. Ms. Lopez wants to place a pile of buttons in the grass halfway between the swings and the merry-go-round. Describe how you can determine the halfway point between the two locations.





8. How far, in yards, from the swings and the merry-go-round will the pile of buttons be? Plot the location of the buttons on the coordinate plane. Round your answer to the nearest tenth if necessary.

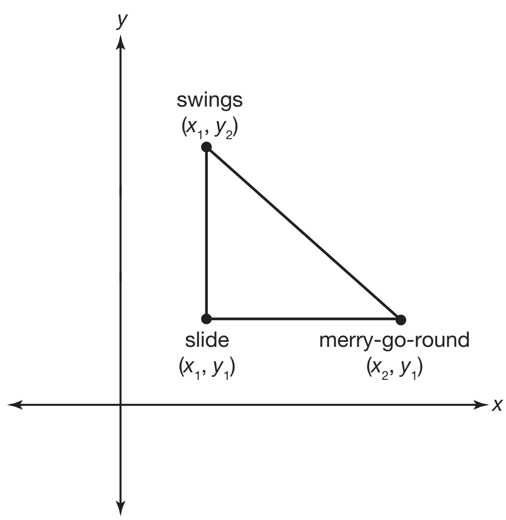
12

9. Verify your solution for Question 8.

   a. Use the Distance Formula to determine whether your answer in Question 8 is correct by calculating the distance between the buttons and the swings. Show all your work.

   b. Would it have mattered if you verified your answer by calculating the distance between the buttons and the merry-go-round? Explain your reasoning.

Suppose the slide, the swings, and the merry-go-round were at different locations. You can generalize their locations by using x_1, x_2, y_1, and y_2, and then solve for the location between each.



10. Use the diagram to describe these distances.

a. the vertical distance from the *x*-axis to the slide

b. the distance from the slide to the swings

c. half the distance from the slide to the swings

d. the vertical distance from the *x*-axis to the slide plus half the distance from the slide to the swings

11. Simplify the expression from Question 10, part (d).

12. Use the diagram to describe these distances.

a. the horizontal distance from the *y*-axis to the slide

b. the distance from the slide to the merry-go-round

c. half the distance from the slide to the merry-go-round

d. the horizontal distance from the *y*-axis to the slide plus half the distance from the slide to the merry-go-round

13. Simplify the expression from Question 12, part (d).

The coordinates of the points that you determined in Questions 11 and 13 are **midpoints,** or points that are exactly halfway between two given points. The calculations you performed can be summarized in the *Midpoint Formula.*

The **Midpoint Formula** states that if (x_1, y_1) and (x_2, y_2) are two points on the coordinate plane, then the midpoint of the line segment that joins these two points is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

14. Use the Midpoint Formula to determine the midpoint between the swings and the merry-go-round from Question 7.

15. Do you think it matters which point you identify as (x_1, y_1) and which point you identify as (x_2, y_2) when you use the Midpoint Formula? Explain why or why not.

16. Determine the midpoint of each line segment that has the given endpoints. Show all your work.

a. (0, 5) and (4, 3)

b. (8, 2) and (6, 0)

12

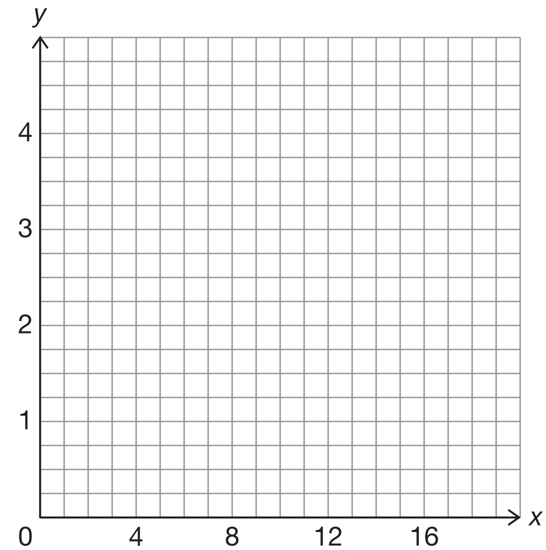
c. (−3, 1) and (9, −7)

d. (−10, 7) and (−4, −7)

PROBLEM 2 Jack's Spare Key

Jack buried a spare key to his house in the backyard in case he ever locked himself out. He remembers burying the key $\frac{1}{4}$ of the way between the back door and an oak tree. The back door is located at point (2, 3), and the oak tree is located at point (12, 3).

1. Plot and label the location of the back door and the oak tree on the coordinate plane.



2. Determine the location of the key.

   a. How would you determine the coordinates of the location of the key?

   b. Determine the coordinates of the location of the key.

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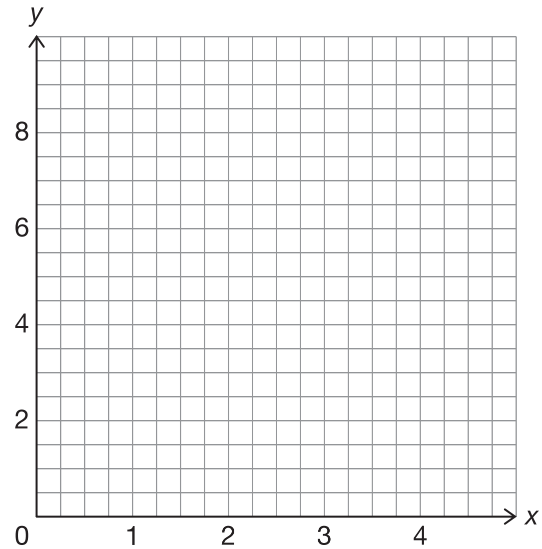
PROBLEM 3 Using the Midpoint Formula Over and Over and Over Again

1. Plot and label points A (0, 4) and B (2, 10) on the coordinate plane.



2. Analyze points A and B.

   a. How would you determine the coordinates of a point that is located halfway between point A and point B?

   b. Determine the coordinates of a point that is located halfway between point A and point B and graph the point.

   c. How would you determine the coordinates of a point that is located $\frac{1}{4}$ of the way between point A and point B?

12

d. Determine the coordinates of a point that is located $\frac{1}{4}$ of the way between point A and point B and graph the point.

e. How would you determine the coordinates of a point that is located $\frac{3}{4}$ of the way between point A and point B?

f. Determine the coordinates of a point that is located $\frac{3}{4}$ of the way between point A and point B and graph the point.

3. Name other fractional divisions of a line segment possible to determine using the Midpoint Formula.

12

PROBLEM 4 Stuck in the Middle

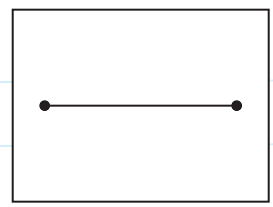
In the previous problem, you located the midpoint of a line segment on the coordinate plane. The lengths of the line segments on the plane are measurable.

In this problem, you will locate the midpoint of a line segment when measurement is not possible. This basic geometric construction used to locate a midpoint of a line segment is called **bisecting a line segment**. When bisecting a line segment, you create a *segment bisector*. A **segment bisector** is a line, line segment, or ray that divides a line segment into two line segments of equal measure, or two congruent line segments.

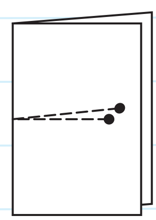
Just as with duplicating a line segment, there are a number of methods to bisect a line segment.

One method is to use tracing paper, which is sometimes known as patty paper.

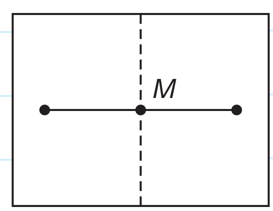
Draw a line segment on the paper.



Fold the paper so that the endpoints of the line segment lie on top of each other.

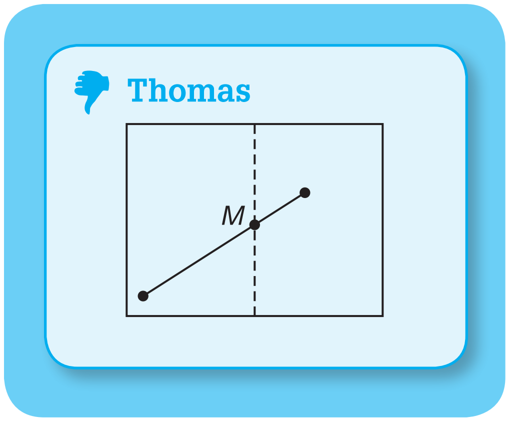


Open the paper. The crease represents the segment bisector, and the midpoint is located where the crease intersects the line segment.



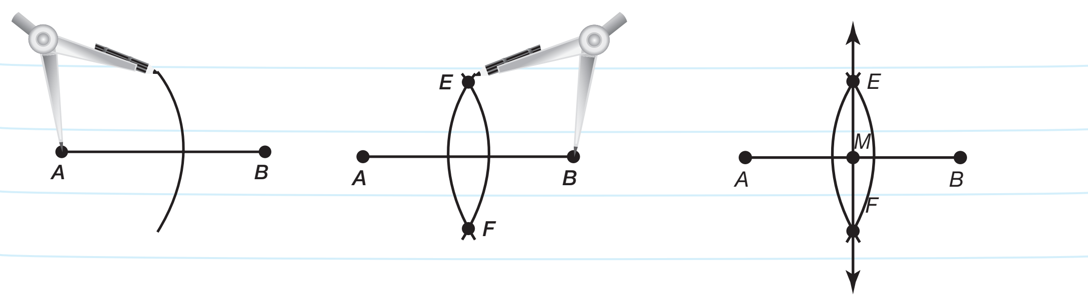
1. Use patty paper to duplicate a line segment. How do you know your bisector and midpoint are accurate?

2. Thomas determined the midpoint of this line segment incorrectly.



Explain what Thomas did incorrectly and how you can tell he is incorrect. Explain how he can correctly determine the midpoint.

Just as with other geometric constructions, a compass and straightedge can be used to determine a segment bisector.



Construct an Arc

Open the radius of the compass to more than half the length of line segment AB. Use endpoint A as the center and construct an arc.

Construct Another Arc

Keep the compass radius and use point B as the center as you construct an arc. Label the points formed by the intersection of the arcs point E and point F.

Construct a Line

Connect points E and F. Line segment EF is the segment bisector of line segment AB. The point M represents the midpoint of $\overline{AB}$.

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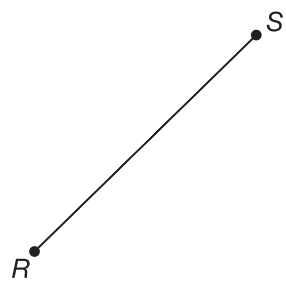
3. Aaron is determining the midpoint of line segment RS. His work is shown.



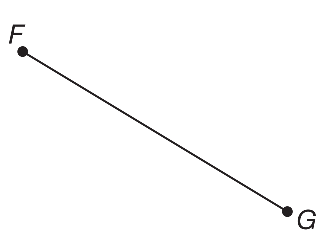
He states that because the arcs do not intersect this line segment does not have a midpoint. Kate disagrees and tells him he drew his arcs incorrectly and they must be redrawn to determine the midpoint. Who is correct? Explain your reasoning.

4. Use construction tools to locate the midpoint of each given line segment. Label each midpoint as M.

a.

b.

c.



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5. When bisecting a line segment, does it make a difference which endpoint is used to draw the first arc? Explain your reasoning.

6. Explain how duplicating a line segment can be used to verify that the midpoint resulting from bisecting the line segment is truly the midpoint of the segment.

Talk the Talk

Two methods for locating the midpoint of a line segment are using the Midpoint Formula and constructing the bisector of a line segment.

1. How are the two methods similar?

2. How are the two methods different?

Be prepared to share your solutions and methods.

12

12.3

It's All About Angles

Translating and Constructing Angles and Angle Bisectors

LEARNING GOALS

In this lesson, you will:

- Translate an angle on the coordinate plane.
- Copy or duplicate an angle by construction.
- Bisect an angle by construction.

KEY TERMS

- angle
- angle bisector

CONSTRUCTIONS

- copying an angle
- duplicating an angle
- bisecting an angle

You may have never thought of it this way, but drawing and geometry are closely linked. Drawing is the process of deliberately arranging lines and curves to create an image. Most drawings have a number of different angles that are created through the intersection of these lines and curves. However, an art movement known as De Stijl limits the types of lines used when drawing to only horizontal and vertical lines. They also limit the colors used to the primary colors. While you may think this sounds restricting, many artists have created many works of art in this style. In fact, an architect even designed a house adhering to the De Stijl principles!

If De Stijl limits the artists to only using horizontal and vertical lines, what types of angles can be created in their art work? What types of angles cannot be created? What might be some challenges with drawing or painting in this style?

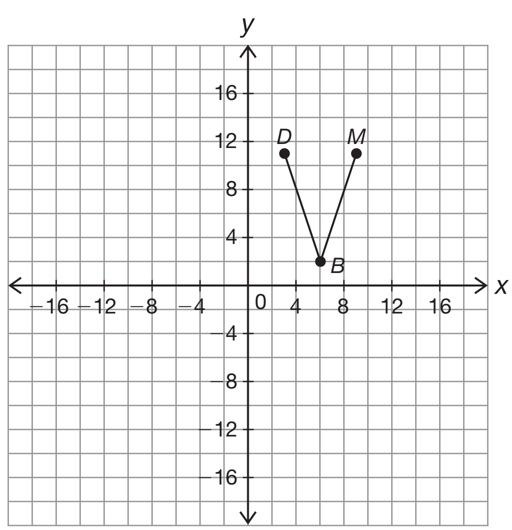
PROBLEM 1 Translating an Angle

In a previous lesson, you practiced translating a line segment on the coordinate plane horizontally or vertically.

1. Describe how to translate a line segment on a coordinate plane.

An **angle** is formed by two rays that share a common endpoint. The sides of the angle are represented by the two rays. Each ray of an angle contains an infinite number of line segments. For the purposes of graphing an angle on the coordinate plane, you will label one line segment on each ray.

2. Analyze angle *DBM*. Describe how you would translate this angle on the coordinate plane.



3. Complete the translations of angle *DBM*.
   a. Horizontally translate $\angle DBM$ 13 units left. Label the image $\angle D'B'M'$.
   b. Vertically translate $\angle D'B'M'$ 15 units down. Label the image $\angle D''B''M''$.
   c. Horizontally translate $\angle D''B''M''$ 13 units right. Label the image $\angle D'''B'''M'''$.
   d. Use the graph to complete the tables.

12

Line Segments	MB	$M'B'$	$M''B''$	$M'''B'''$
Coordinates of Endpoints				

Line Segments	DB	$D'B'$	$D''B''$	$D'''B'''$
Coordinates of Endpoints				

4. Describe how a horizontal translation changes the coordinates of the angle endpoints.

5. Describe how a vertical translation changes the coordinates of the angle endpoints.

6. How many translations must occur for the angle to return to the location of the original figure?

An angle is measured using a protractor, and the measure of an angle is expressed in units called degrees.

7. Measure each angle on the coordinate plane. How do the measures of the images compare to the measures of the pre-images?

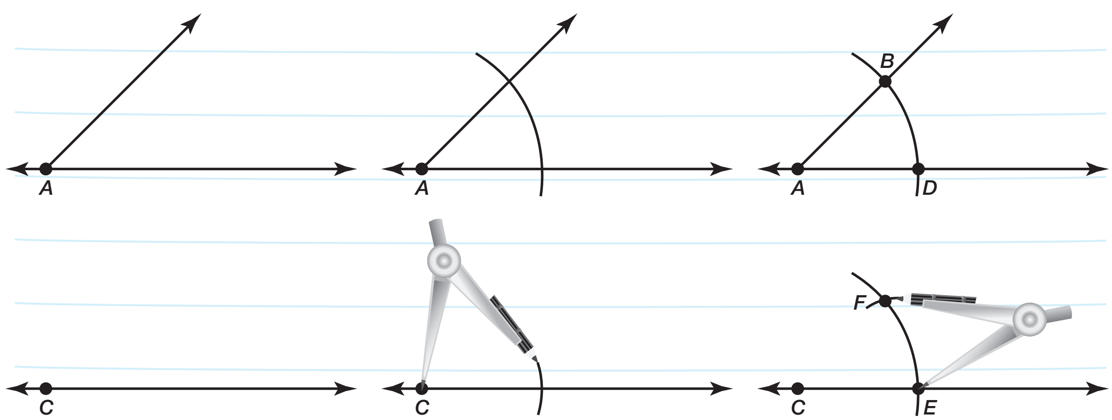
8. Does either a vertical or a horizontal translation of an angle alter the measure of the angle? Explain why or why not.

9. What is the result of moving only one angle endpoint a specified distance or direction? How does this affect the measure of the angle? Is this still considered a translation?

PROBLEM 2 Constructing an Angle

In the previous problem, you translated an angle on the coordinate plane using line segments that were associated with a measurement.

You can also translate an angle not associated with a measurement. This basic geometric construction is called **copying an angle** or **duplicating an angle**. The construction is performed using a compass and a straightedge.



Construct a Starter Line

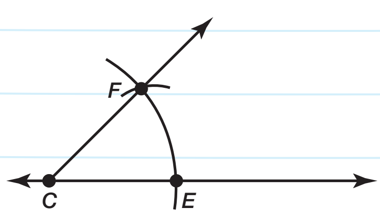
Use a straightedge to construct a starter line. Label point *C* on the new segment.

Construct an Arc

Construct an arc with center *A*. Using the same radius, construct an arc with center *C*.

Construct Another Arc

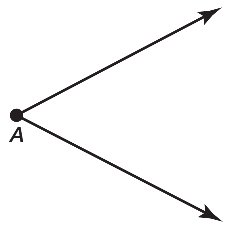
Label points *B*, *D*, and *E*. Construct an arc with radius *BD* and center *E*. Label the intersection *F*.



Construct a Ray

Construct ray *CF*. $\angle BAD \cong \angle FCE$.

1. Construct an angle that is twice the measure of $\angle A$.



2. How is duplicating an angle similar to duplicating a line segment?
How is it different?

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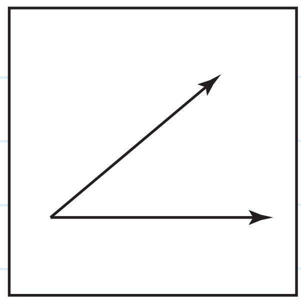
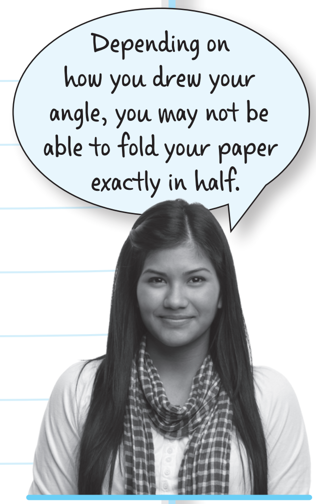
PROBLEM 3 Bisecting an Angle

Just as line segments can be bisected, angles can be bisected too. If a ray is drawn through the vertex of an angle and divides the angle into two angles of equal measure, or two congruent angles, the ray is called an **angle bisector**. The construction used to create an angle bisector is called **bisecting an angle**.

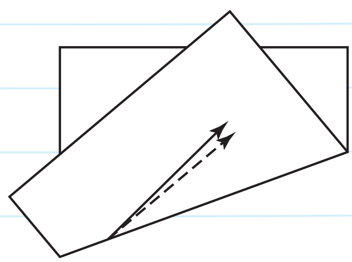
One way to bisect an angle is using patty paper.

Draw an angle on the paper.

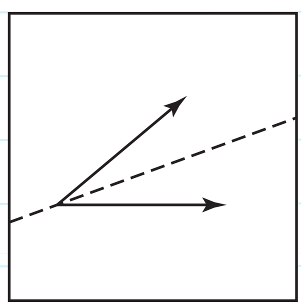




Fold the paper so the rays lie on top of each other. The fold should pass through the vertex of the angle.



Open the paper. The crease represents the angle bisector.



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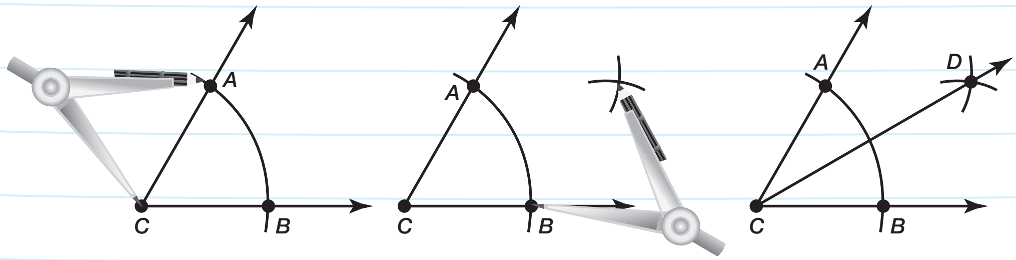
1. Angela states that as long as the crease goes through the vertex it is an angle bisector. Is she correct? Why or why not?

You can also bisect an angle using a compass and a straightedge.



Construct an Arc

Place the compass at *C*. Construct an arc that intersects both sides of the angle. Label the intersections *A* and *B*.

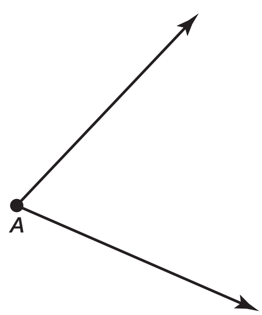
Construct Another Arc

Place the compass at *A*. Construct an arc. Then, place the compass point at *B*. Using the same radius, construct another arc.

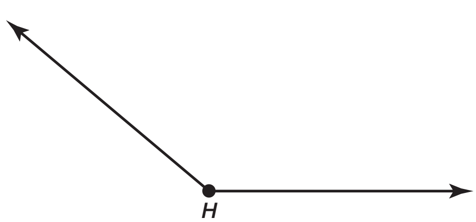
Construct a Ray

Label the intersection of the two arcs *D*. Use a straightedge to construct a ray through *C* and *D*. Ray *CD* bisects $\angle C$.

2. Construct the bisector of $\angle A$.

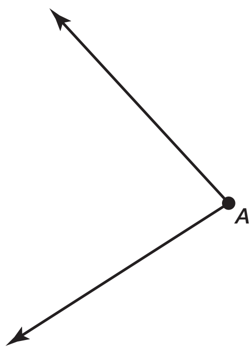


3. Construct an angle that is one-fourth the measure of $\angle H$.



4. Describe how to construct an angle that is one-eighth the measure of angle *H*.

5. Use a compass and straightedge to show that the two angles formed by the angle bisector of angle *A* are congruent. Describe each step.



Talk the Talk

Translating an angle on the coordinate plane using coordinates and translating an angle by construction using construction tools both preserve the measure of the angle.

1. How are the two methods of translation similar?

12

2. How are the two methods of translation different?

Be prepared to share your solutions and methods.

Did You Find a Parking Space?

12.4

Parallel and Perpendicular Lines on the Coordinate Plane

LEARNING GOALS

In this lesson, you will:

- Determine whether lines are parallel.
- Identify and write the equations of lines parallel to given lines.
- Determine whether lines are perpendicular.
- Identify and write the equations of lines perpendicular to given lines.
- Identify and write the equations of horizontal and vertical lines.
- Calculate the distance between a line and a point not on the line.

KEY TERM

- point-slope form

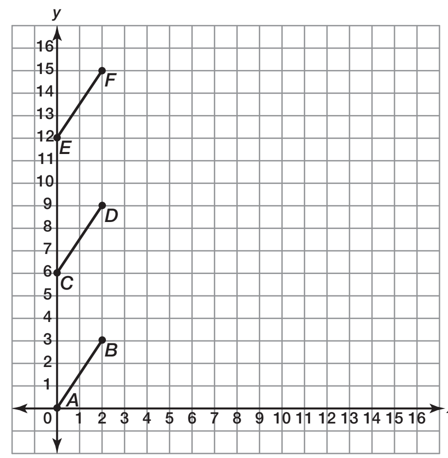
They seem simple enough, but parking lots require a great deal of planning. Parking lots are designed by transportation engineers who use technology and science to plan, design, operate, and manage facilities for any mode of transportation. During the planning stage of a parking lot, these engineers must keep in mind the needs of the facility that will use the parking lot as well as the needs of the drivers. They must think about the entrances and exits as well as the surrounding streets and their traffic flow. Even the weather must be taken into account if the lot is being built somewhere with heavy rain or snow!

Only thinking about the cars and their drivers, what needs might affect an engineer's plans? What would make a parking lot "good" or "bad"? Can you think of anything else that might affect the planning of a parking lot other than the things mentioned above?

PROBLEM 1 Parking Spaces

Large parking lots have line segments painted to mark the locations where vehicles are supposed to park. The layout of these line segments must be considered carefully so that there is enough room for the vehicles to move and park in the lot without other vehicles being damaged.

The line segments shown model parking spaces in a parking lot. One grid square represents one square meter.



1. What do you notice about the line segments that form the parking spaces?

2. What is the vertical distance between $\overline{AB}$ and $\overline{CD}$ and between $\overline{CD}$ and $\overline{EF}$?

3. Carefully extend $\overline{AB}$ to create line p, extend $\overline{CD}$ to create line q, and extend $\overline{EF}$ to create line r.

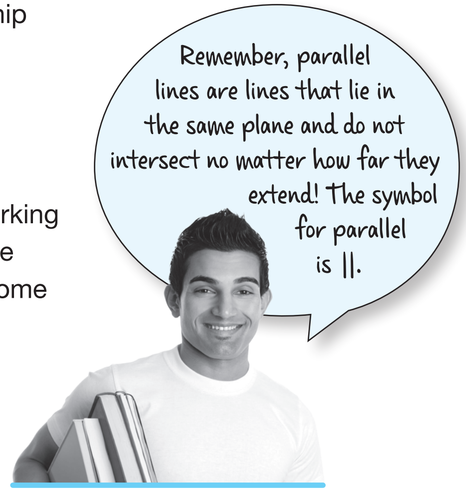
4. Use the graph to identify the slope of each line. What do you notice?

12

The **point-slope form** of the equation of the line that passes through (x_1, y_1) and has slope m is $y - y_1 = m(x - x_1)$.

5. Use the point-slope form to write the equations of lines p, q, and r. Then write the equations in slope-intercept form.

6. What do the y-intercepts tell you about the relationship between these lines in the problem situation?



7. If you were to draw $\overline{GH}$ above $\overline{EF}$ to form another parking space, predict what the slope and equation of the line will be without graphing the new line. How did you come to this conclusion?

8. In the *Parking Spaces* problem, all the slopes were equal and the y-intercepts were all multiples of the same number.

   a. Are the slopes of parallel lines on a coordinate plane always equal? Explain your reasoning.

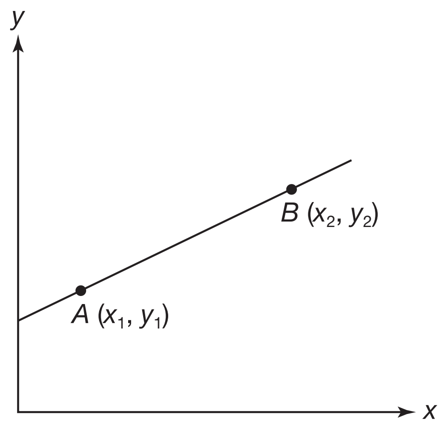
   b. Are the y-intercepts of parallel lines on a coordinate plane always a multiple of the same number? Explain your reasoning.

9. Write equations for three lines that are parallel to the line given by $y = -2x + 4$. Explain how you determined your answers.

10. Write an equation for the line that is parallel to the line given by $y = 5x + 3$ and passes through the point (4, 0). Explain how you determined your answer.

11. Without graphing the equations, predict whether the lines given by $y - 2x = 5$ and $2x - y = 4$ are parallel.

12. Consider the graph shown.



a. Use the graph to translate line segment *AB* *a* units up.

b. Identify the *x*- and *y*-coordinates of each corresponding point on the image.

c. Use the slope formula to calculate the slope of the pre-image.

12

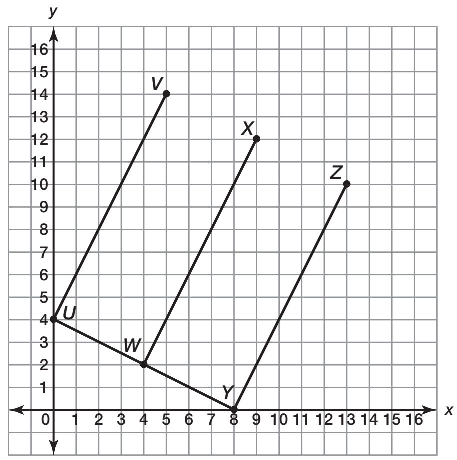
d. Use the slope formula to calculate the slope of the image.

e. How does the slope of the image compare to the slope of the pre-image?

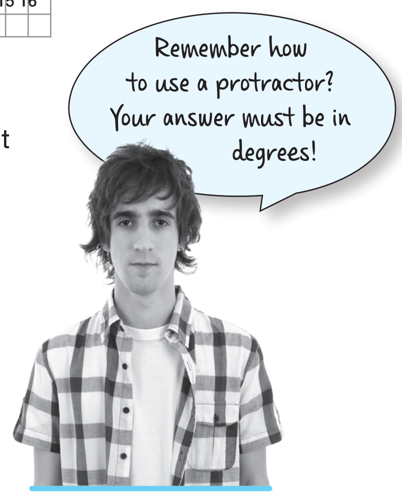
f. How would you describe the relationship between the graph of the image and the graph of the pre-image?

PROBLEM 2 More Parking Spaces

The line segments shown represent parking spaces in a truck parking lot. One grid square represents one square meter.



1. Use a protractor to determine the measures of $\angle VUW$, $\angle XWY$, and $\angle ZYW$. What similarity do you notice about the angles?



2. Carefully extend $\overline{UY}$ to create line p, extend $\overline{UV}$ to create line q, extend $\overline{WX}$ to create line r, and extend $\overline{YZ}$ to create line s on the coordinate plane.

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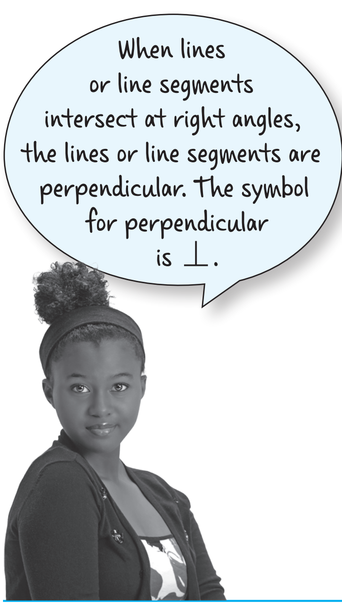


3. Determine whether the lines given are perpendicular or parallel.

   a. q, r, s

   b. p and q

   c. p and r

   d. p and s

4. Predict how the slopes of the lines will compare. Do not actually calculate the slopes of the lines for your prediction.

5. Use the graph and the lines you drew to determine the slopes of lines p, q, r, and s.

6. Determine the product of the slopes of two perpendicular lines.

7. Describe the difference between the slopes of two parallel lines and the slopes of two perpendicular lines.

When the product of two numbers is 1, the numbers are reciprocals of one another. When the product of two numbers is -1, the numbers are negative reciprocals of one another. So the slopes of perpendicular lines are negative reciprocals of each other.

8. Do you think that the y-intercepts of perpendicular lines tell you anything about the relationship between the perpendicular lines? Explain your reasoning.

12

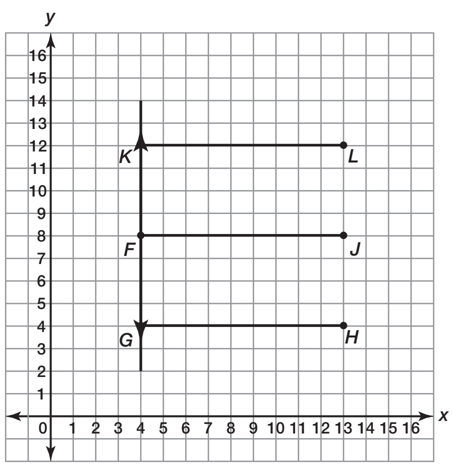
9. Write equations for three lines that are perpendicular to the line given by $y = -2x + 4$. Explain how you determined your answers.

10. Write an equation for the line that is perpendicular to the line given by $y = 5x + 3$ and passes through the point (4, 0). Show all your work and explain how you determined your answer.

11. Without graphing the equations, determine whether the lines given by $y + 2x = 5$ and $2x - y = 4$ are perpendicular. Explain how you determined your answer.

PROBLEM 3 Horizontal and Vertical

Consider the graph shown.



1. Carefully extend $\overline{GK}$ to create line p, extend $\overline{GH}$ to create line q, extend $\overline{FJ}$ to create line r, and extend $\overline{KL}$ to create line s.

12

2. Consider the three horizontal lines you drew for Question 1. For any horizontal line, if x increases by one unit, by how many units does y change?

3. What is the slope of any horizontal line? Explain your reasoning.

4. Consider the vertical line you drew in Question 1. Suppose that y increases by one unit. By how many units does x change?

5. What is the rise divided by the run? Does this make any sense? Explain why or why not.

6. Determine whether the statements are always, sometimes, or never true. Explain your reasoning.

   a. Vertical lines are always parallel.

   b. Horizontal lines are always parallel.

12

7. Describe the relationship between any vertical line and any horizontal line.

8. Write an equation for a horizontal line and an equation for a vertical line that pass through the point $(2, -1)$.

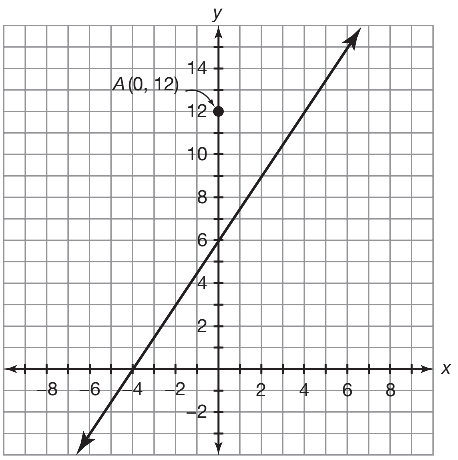
9. Write an equation for a line that is perpendicular to the line given by $x = 5$ and passes through the point $(1, 0)$.

10. Write an equation for a line that is perpendicular to the line given by $y = -2$ and passes through the point $(5, 6)$.

PROBLEM 4 Distance Between Lines and Points

1. Sketch a line and a point not on the line. Describe the shortest distance between the point and the line.

2. The equation of the line shown on the coordinate plane is $f(x) = \frac{3}{2}x + 6$. Draw the shortest segment between the line and the point A (0, 12). Label the point where the segment intersects $f(x)$ as point B.



3. What information do you need in order to calculate the length of $\overline{AB}$ using the Distance Formula?

4. How can you calculate the intersection point of $\overline{AB}$ and the line $f(x) = \frac{3}{2}x + 6$ algebraically?

5. Write an equation for $\overline{AB}$.

12

6. Calculate the point of intersection of $\overline{AB}$ and the line $f(x) = \frac{3}{2}x + 6$.

7. Calculate the length of $\overline{AB}$.

8. What is the distance from the point (0, 12) to the line $f(x) = \frac{3}{2}x + 6$?

12

Be prepared to share your solutions and methods.

Making Copies—Just as Perfect as the Original!

12.5

Constructing Perpendicular Lines, Parallel Lines, and Polygons

OBJECTIVES

In this lesson, you will:

- Construct a perpendicular line to a given line through a point on the line.
- Construct a perpendicular line to a given line through a point not on the line.
- Construct a parallel line to a given line through a point not on the line.
- Construct an equilateral triangle given the length of one side of the triangle.
- Construct an isosceles triangle given the length of one side of the triangle.
- Construct a square given the perimeter (as the length of a given line segment).
- Construct a rectangle that is not a square given the perimeter (as the length of a given line segment).

There's an old saying that you might have heard before: "They broke the mold when they made me!" A person says this to imply that they are unique. Of course, humans do not come from molds, but there are plenty of things that do.

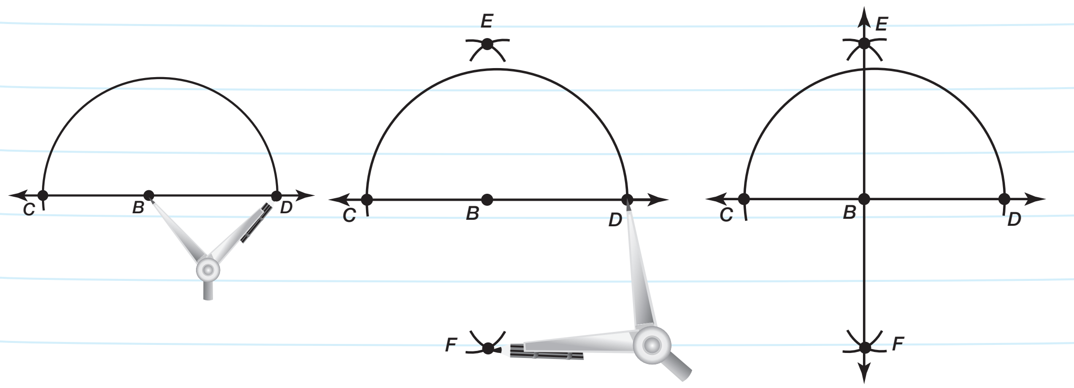
For example, take a look at a dime if you have one handy. Besides some tarnish on the coin and the year the coin was produced, it is identical to just about every other dime out there. Creating and duplicating a coin a few billion times is quite a process involving designing the coin, creating multiple molds (and negatives of the molds), cutting the design onto metal, and on and on.

Can you think of any times when the "original" might be more important than a duplicate? Can you think of any examples where the "original" product might be more expensive than a generic brand of the same product?

PROBLEM 1 Constructing Perpendicular Lines

In a previous lesson, you practiced bisecting a line segment and locating a midpoint of a line segment by construction. In fact, you were actually constructing a line segment perpendicular to the original line segment during the construction.

Follow the steps to construct a perpendicular line through a point on the line.



Construct an Arc

Use B as the center and construct an arc. Label the intersections points C and D.

Construct Other Arcs

Open the compass radius. Use C and D as centers and construct arcs above and below the line. Label the intersection points E and F.

Construct a Line

Use a straightedge to connect points E and F. Line EF is perpendicular to line CD.

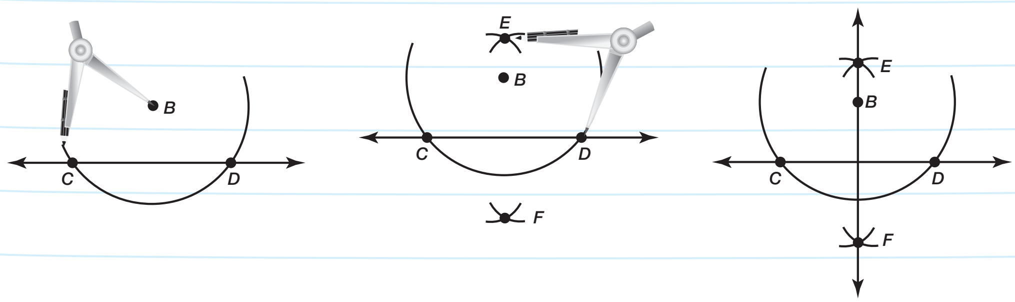
12

1. Construct a line perpendicular to the given line through point P.

2. How is constructing a segment bisector and constructing a perpendicular line through a point on a line different?

3. Do you think that you can only construct a perpendicular line through a point that is on a line? Why or why not?

It is also possible to construct a perpendicular line through a point not on a line.



Construct an Arc

Use *B* as the center and construct an arc. Label the intersection points *C* and *D*.

Construct Other Arcs

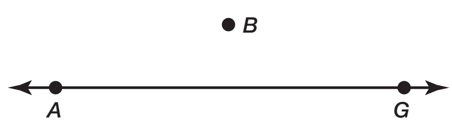
Open the compass radius. Use *C* and *D* as centers and construct arcs above and below the line. Label the intersection points *E* and *F*.

Construct a Line

Use a straightedge to connect points *E* and *F*. Line *EF* is perpendicular to line *CD*.

4. Amos claims that it is only possible to construct a perpendicular line through horizontal and vertical lines because the intersection of the points must be right angles. Loren claims that a perpendicular line can be constructed through any line and any point on or not on the line. Who is correct? Correct the rationale of the student who is *not* correct.

5. Construct a line perpendicular to $\overleftrightarrow{AG}$ through point B.



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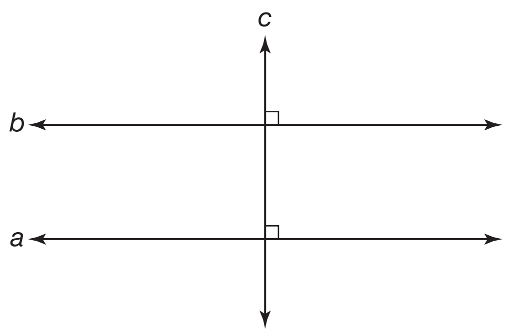
6. How is the construction of a perpendicular line through a point on a line different from the construction of a perpendicular line through a point not on the line?

PROBLEM 2 Constructing Parallel Lines

You can construct a line parallel to a given line. Of course, to ensure that the constructed line is parallel, you must use a perpendicular line.

1. Analyze the figure shown.



Describe the relationship between the lines given.

a. *a* and *c*

b. *b* and *c*

c. *a* and *b*

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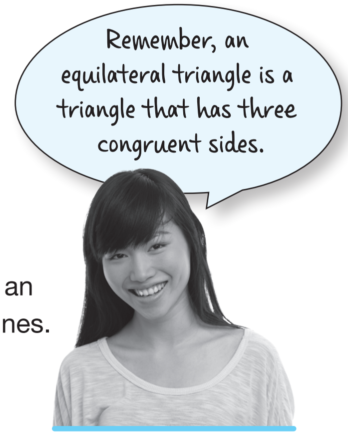
2. Use line d to construct line e parallel to line d. Then, describe the steps you performed for the construction.

PROBLEM 3 Constructing an Equilateral Triangle



In the rest of this lesson, you will construct an equilateral triangle, an isosceles triangle, a square, and a rectangle that is not a square. To perform the constructions, use only a compass and straightedge and rely on the basic geometric constructions you have previously practiced such as duplicating a line segment, duplicating an angle, bisecting a line segment, bisecting an angle, constructing perpendicular lines, and constructing parallel lines.

The length of one side of an equilateral triangle is shown.

1. What do you know about the other two sides of the equilateral triangle you will construct given the line segment shown?

2. Write a paragraph explaining how you will construct this equilateral triangle.

3. Construct the equilateral triangle using the starter line given.

4. Sophie claims that she can construct an equilateral triangle by duplicating the line segment three times and having the endpoints of all three line segments intersect. Roberto thinks that Sophie's method will not result in an equilateral triangle. Who is correct? Explain why the incorrect student's rationale is not correct.

PROBLEM 4 Constructing an Isosceles Triangle

The length of one side of an isosceles triangle that is not an equilateral triangle is shown.

1. Write a paragraph to explain how you will construct this isosceles triangle that is not an equilateral triangle.

2. Construct an isosceles triangle that is *not* an equilateral triangle using the starter line given below.

3. Explain how you know your construction resulted in an isosceles triangle that is not an equilateral triangle.

PROBLEM 5 Constructing a Square Given the Perimeter

Now you will construct a square using a given perimeter.

1. The perimeter of a square is shown by $\overline{AB}$.

   a. Write a paragraph to explain how you will construct this square.

   b. Construct the square.

PROBLEM 6 Constructing a Rectangle Given the Perimeter

1. The perimeter of a rectangle is shown by $\overline{AB}$.

a. Write a paragraph to explain how you will construct this rectangle that is not a square.

b. Construct the rectangle that is not a square.

Be prepared to share your solutions and methods.

Chapter 12 Summary

KEY TERMS

- Distance Formula (12.1)
- transformation (12.1)
- rigid motion (12.1)
- translation (12.1)
- image (12.1)
- pre-image (12.1)
- arc (12.1)
- congruent line segments (12.1)
- congruent (12.1)
- midpoint (12.2)
- Midpoint Formula (12.2)
- segment bisector (12.2)
- angle (12.3)
- angle bisector (12.3)
- point-slope form (12.4)

CONSTRUCTIONS

- copying a line segment (12.1)
- duplicating a line segment (12.1)
- bisecting a line segment (12.2)
- copying an angle (12.3)
- duplicating an angle (12.3)
- bisecting an angle (12.3)

12.1 Applying the Distance Formula

The Distance Formula can be used to calculate the distance between two points on the coordinate plane. The Distance Formula states that if (x_1, y_1) and (x_2, y_2) are two points on the coordinate plane, then the distance d between (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

12

Example

Calculate the distance between the points $(3, -2)$ and $(-5, 1)$.

$x_1 = 3, y_1 = -2, x_2 = -5, y_2 = 1$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d = \sqrt{(-5 - 3)^2 + [1 - (-2)]^2}$

$d = \sqrt{(-8)^2 + (3)^2}$

$d = \sqrt{64 + 9}$

$d = \sqrt{73}$

$d \approx 8.5$

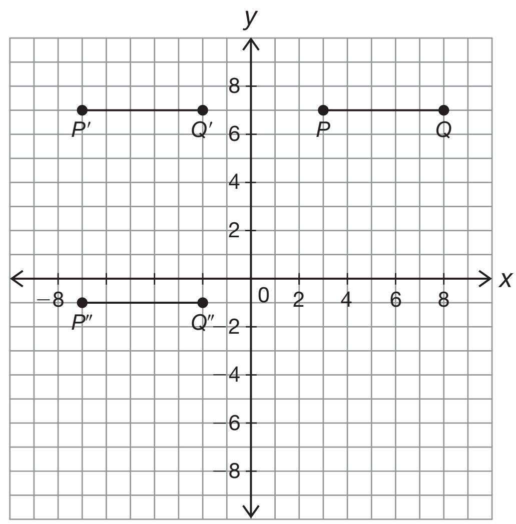
The distance between the points $(3, -2)$ and $(-5, 1)$ is $\sqrt{73}$ units, or approximately 8.5 units.

12.1 Translating Line Segments on the Coordinate Plane

A translation is a rigid motion that slides each point of a figure the same distance and direction. A horizontal translation of a line segment on the coordinate plane changes the *x*-coordinates of both endpoints while leaving the *y*-coordinates the same. A vertical translation changes the *y*-coordinates of both endpoints while leaving the *x*-coordinates the same.

Example

Line segment *PQ* is translated horizontally 10 units to the left to create *P′Q′*. Line segment *P′Q′* is translated vertically 8 units down to create line segment *P″Q″*.



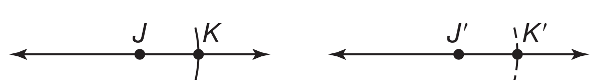
Line Segment	*PQ*	*P′Q′*	*P″Q″*
Coordinates of Endpoints	(3, 7) (8, 7)	(−7, 7) (−2, 7)	(−7, −1) (−2, −1)

The lengths of the images and the pre-images remain the same after each translation.

12.1 Duplicating a Line Using Construction Tools

A straightedge and compass can be used to duplicate a line.

Example



Line segment *JK* can be duplicated using a straightedge and compass by drawing a starter line and then copying a line segment that is the same length as $\overline{JK}$.

12.2 Applying the Midpoint Formula

A midpoint is a point that is exactly halfway between two given points. The Midpoint Formula can be used to calculate the coordinates of a midpoint. The Midpoint Formula states that if (x_1, y_1) and (x_2, y_2) are two points on the coordinate plane, then the midpoint of the line segment that joins these two points is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example

Calculate the midpoint of a line segment with the endpoints $(-8, -3)$ and $(4, 6)$.

$x_1 = -8, y_1 = -3, x_2 = 4, y_2 = 6$

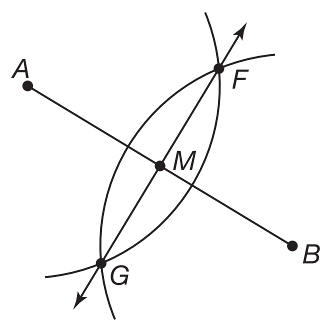
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-8 + 4}{2}, \frac{-3 + 6}{2}\right)$$
$$= \left(\frac{-4}{2}, \frac{3}{2}\right)$$
$$= \left(-2, \frac{3}{2}\right)$$

The midpoint of the line segment is $\left(-2, \frac{3}{2}\right)$.

12.2 Bisecting a Line Segment Using Construction Tools

Construction tools can be used to bisect a line segment.

Example



Open the radius of the compass to more than half the length of the original line segment. Construct an arc using one endpoint as the center. Keeping the compass at the same radius, construct an arc using the other endpoint as center. Label and connect the points created by the intersection of the arcs. Line segment *FG* bisects line segment *AB*.

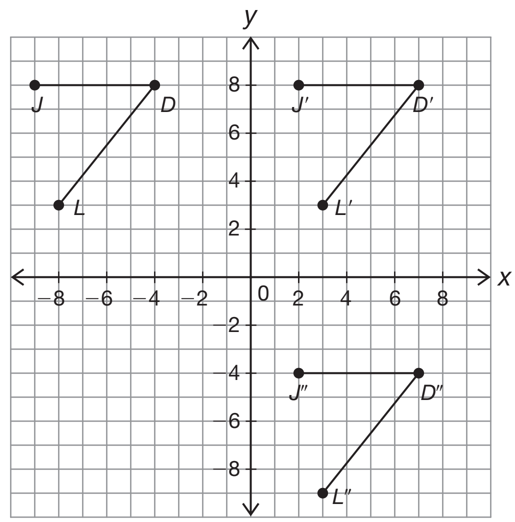
12

12.3 Translating an Angle on the Coordinate Plane

Translating an angle on the coordinate plane is a rigid motion that slides the angle, either horizontally or vertically, on the coordinate plane. Because it is a rigid motion, the angle measures of the image and the pre-image are the same. Horizontal translations only impact the *x*-coordinates of the endpoints; vertical translations only impact the *y*-coordinates of the endpoints.

Example

Angle *JDL* is translated horizontally 11 units right to form angle *J′D′L′*. Angle *J′D′L′* is translated vertically 12 units down to create angle *J″D″L″*.



Line Segment	*JD*	*J′D′*	*J″D″*
Coordinates of Endpoints	(−9, 8) (−4, 8)	(2, 8) (7, 8)	(2, −4) (7, −4)

Line Segment	*DL*	*D′L′*	*D″L″*
Coordinates of Endpoints	(−4, 8) (−8, 3)	(7, 8) (3, 3)	(7, −4) (3, −9)

The measure of the angle images and pre-images remain the same after each translation.

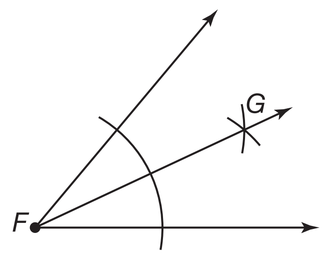
12

12.3 Bisecting an Angle Using Construction Tools

An angle bisector is a ray drawn through the vertex of an angle that divides the angle into two angles of equal measure.

Example

Angle *F* can be bisected using construction tools.



Place the compass on the vertex of the angle. Construct an arc that intersects both sides of the angle. Place the compass at one of the intersection points and construct an arc, then using the same radius of the compass construct an arc using the other intersection point. Construct a ray connecting the vertex to the intersection of the arcs. Ray *FG* bisects angle *F*.

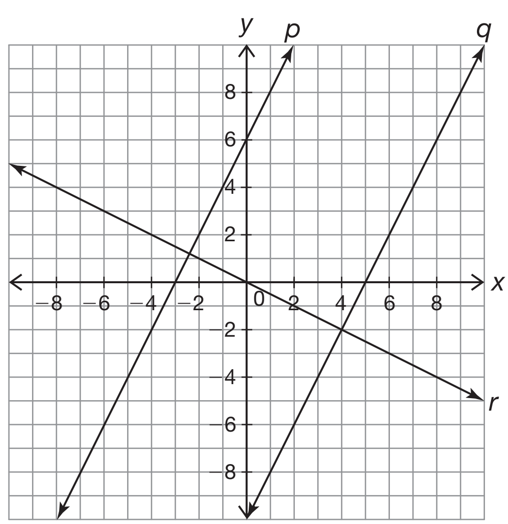
12.4 Determining Whether Lines Are Parallel or Perpendicular

The slopes of parallel lines are equal. The slopes of perpendicular lines are negative reciprocals of each other. When the product of two numbers is -1, the numbers are negative reciprocals of each other.

Example

The equation of line p is $y = 2x + 6$, the equation of line q is $y = 2x - 10$, and the equation of line r is $y = -\frac{1}{2}x$. The slopes of lines p and q are equal, so lines p and q are parallel.

The slopes of lines p and r are negative reciprocals of each other, so lines p and r are perpendicular. Also, the slopes of lines q and r are negative reciprocals of each other, so lines q and r are also perpendicular.



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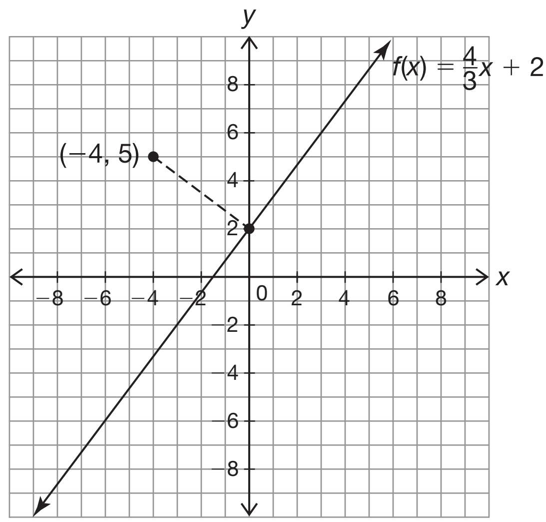
12.4 Determining the Distance Between Lines and Points

The shortest distance between any line and a point not on that line is the length of the perpendicular segment drawn from the point to the line. The shortest distance between a line and a point not on the line can be determined using the equation of the perpendicular segment drawn from the point to the line, the equation of the original line, and the Distance Formula.

Example

Calculate the distance between the line given by the equation $f(x) = \frac{4}{3}x + 2$ and the point $(-4, 5)$.



Equation of perpendicular segment:

$y = mx + b$

$5 = -\frac{3}{4}(-4) + b$

$5 = 3 + b$

$2 = b$

$y = -\frac{3}{4}x + 2$

Point of intersection:

$\frac{4}{3}x + 2 = -\frac{3}{4}x + 2$

$16x = -9x$

$25x = 0$

$x = 0$

$y = \frac{4}{3}(0) + 2$

$y = 2$

$(0, 2)$

Distance between point of intersection and given point:

$x_1 = 0, y_1 = 2, x_2 = -4, y_2 = 5$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d = \sqrt{(-4 - 0)^2 + (5 - 2)^2}$

$d = \sqrt{(-4)^2 + (3)^2}$

$d = \sqrt{16 + 9}$

$d = \sqrt{25}$

$d = 5$

The distance between the line given by the equation $f(x) = \frac{4}{3}x + 2$ and the point $(-4, 5)$ is 5 units.

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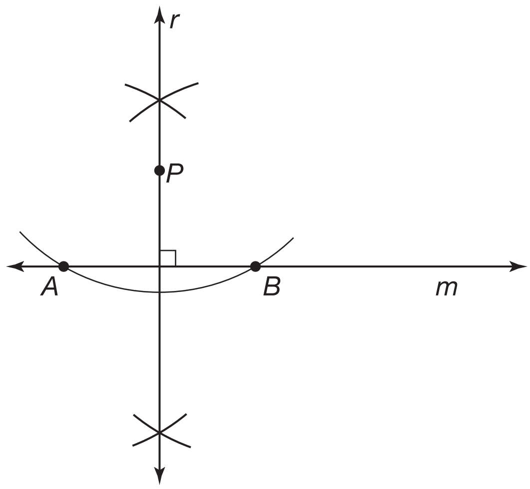
12.5 Constructing Perpendicular Lines

Perpendicular lines can be constructed through a given point using construction tools.

Example



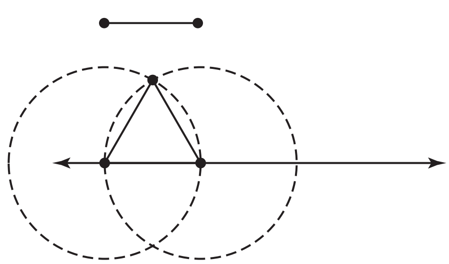
Use the given point *P* as the center and construct an arc that passes through the given line. Open the compass radius. Construct an arc above and below the given line using one of the intersection points just created. Keeping the radius the same, construct an arc above and below the given line using the other intersection point. Connect the intersection points of the arcs which should also pass through the given point. Line *r* is perpendicular to line *m*.

12.5 Constructing Equilateral Triangles

Equilateral triangles have 3 congruent sides. Construction tools can be used to construct an equilateral triangle given the length of one side.

Example

Construct an equilateral triangle with the side length shown.



Construct a starter line and duplicate the given segment onto the starter line. Construct a circle using an endpoint of the line segment as the center. Then construct another circle using the other endpoint as the center. Connect the point of intersection of the circles to each endpoint using line segments.

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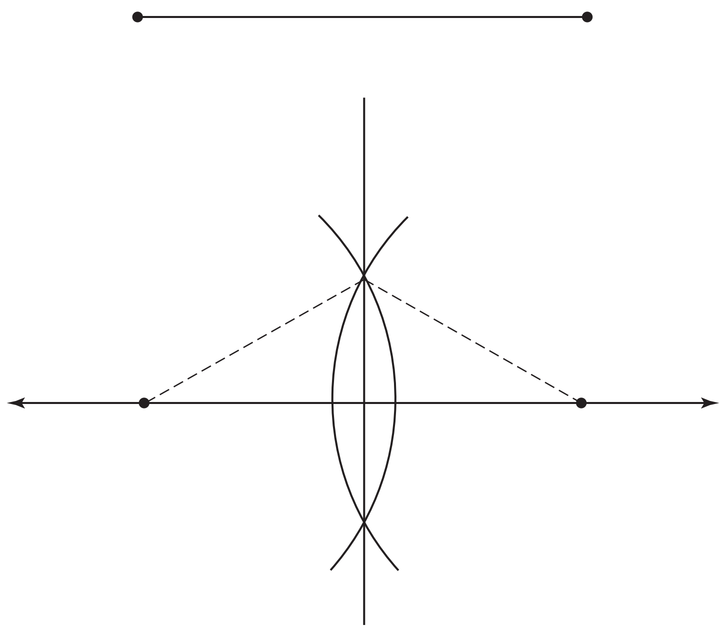
12.5 Constructing Isosceles Triangles

An isosceles triangle is a triangle that has at least two sides of equal length.

Example

Construct an isosceles triangle with the side length shown.



Construct a starter line and duplicate the given line segment. Then construct a perpendicular bisector through the line segment. Connect the endpoints of each line segment to a point on the bisector.

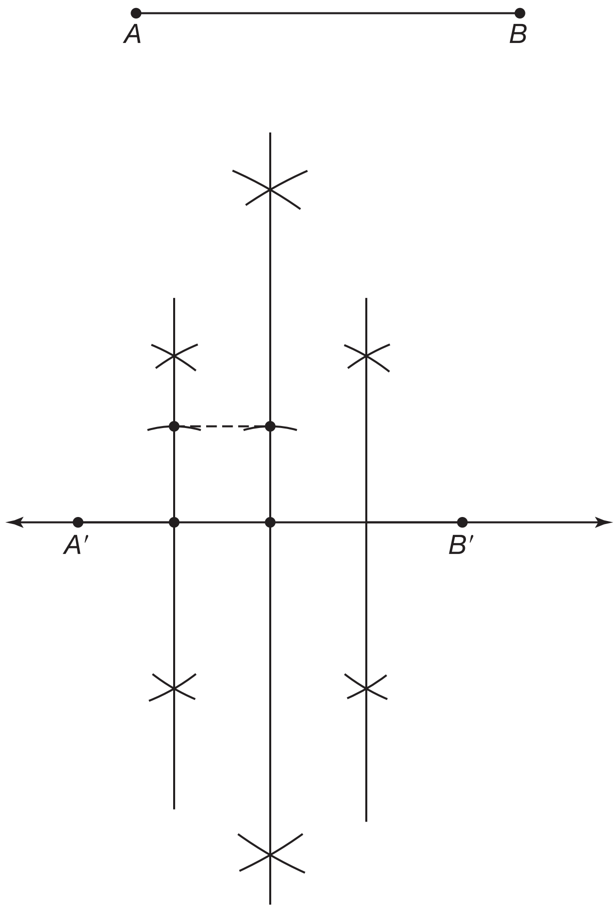
12

12.5 Constructing Squares

A square can be constructed using construction tools.

Example

Construct a square using the perimeter given.



Construct a starter line and duplicate the given perimeter. Bisect the line segment using a perpendicular bisector. Then bisect each of the created line segments to create 4 line segments of equal length. Duplicate one of the line segments along two perpendicular bisectors to create the height of the square. Connect the two endpoints of the line segments representing the height to complete the square.

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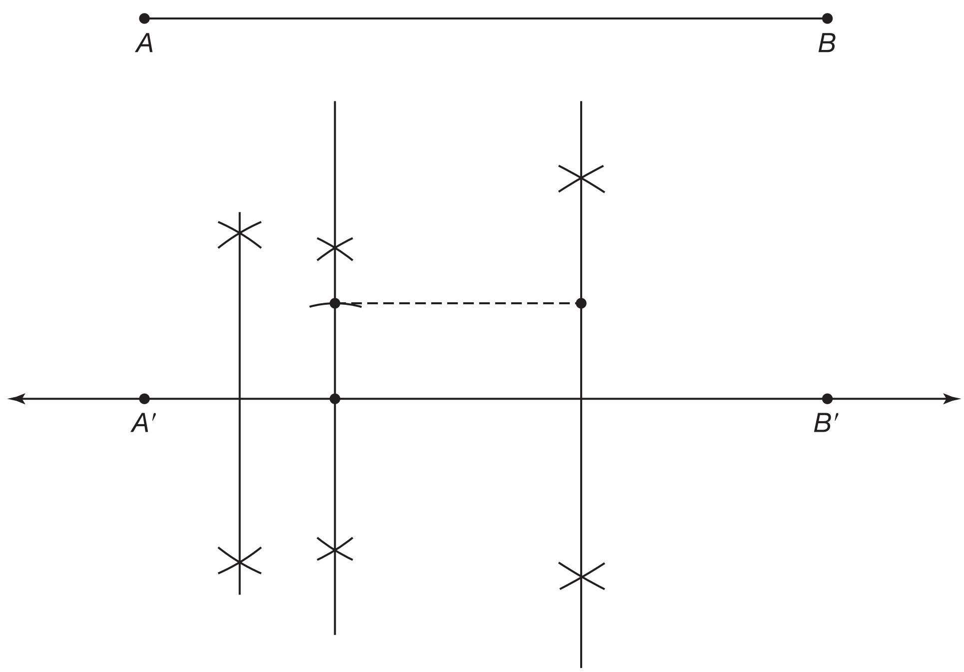
12.5 Constructing Rectangles That Are Not Squares

A rectangle can be constructed in a similar method to constructing a square using a given perimeter of the rectangle.

Example

Construct a rectangle using the perimeter given.



Construct a starter line and duplicate the given perimeter. Place a point anywhere on the line segment except in the middle dividing the line segment into two unequal line segments. Then draw perpendicular bisectors through each of the line segments to create four line segments. Choose one of the line segments to use as the base of the rectangle. Duplicate another line segment that is not the same size as the base on two of the perpendicular bisectors to use as the height of the rectangle. Finally, connect the endpoints of the line segments representing the height to create a rectangle.

12