Congruence Through Transformations

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Slide, Flip, Turn: The Latest Dance Craze?

13.1

Translating, Rotating, and Reflecting Geometric Figures

OBJECTIVES

In this lesson, you will:

- Translate geometric figures on a coordinate plane.
- Rotate geometric figures on a coordinate plane.
- Reflect geometric figures on a coordinate plane.

KEY TERMS

- rotation
- point of rotation
- angle of rotation
- reflection
- line of reflection

Did you know that most textbooks are translated from English into at least one other language, usually Spanish? And in some school districts, general memos and letters to parents may be translated to up to 5 different languages! Of course, *translating* a language means something completely different from *translating* in geometry.

The same can be said for reflection. A "reflection pool" is a place where one can "reflect" on one's thoughts, while also admiring reflections in the pool of still water.

How about rotation? What do you think the term *rotation* means in geometry? Is this different from its meaning in common language?

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To begin this chapter, cut out the figure shown.

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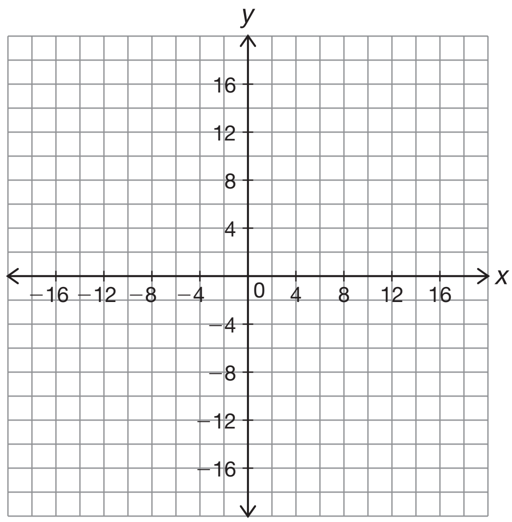
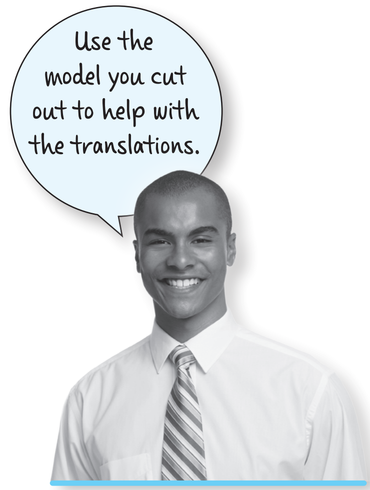
PROBLEM 1 Translating Geometric Figures on the Coordinate Plane

1. Graph trapezoid *ABCD* by plotting the points *A* (3, 9), *B* (3, 4), *C* (11, 4), and *D* (11, 10).





2. Translate the trapezoid on the coordinate plane. Graph the image and record the vertex coordinates in the table.

a. Translate trapezoid *ABCD* 15 units to the left to form trapezoid *A′B′C′D′*.

b. Translate trapezoid *ABCD* 12 units down to form trapezoid *A″B″C″D″*.

Trapezoid *ABCD* (coordinates)	Trapezoid *A′B′C′D′* (coordinates)	Trapezoid *A″B″C″D″* (coordinates)
A (3, 9)		
B (3, 4)		
C (11, 4)		
D (11, 10)		

Let's consider translations without graphing.

3. The vertices of parallelogram *DEFG* are *D* (−9, 7), *E* (−12, 2), *F* (−3, 2), and *G* (0, 7).

a. Determine the vertex coordinates of image $D'E'F'G'$ if parallelogram *DEFG* is translated 14 units down.

b. How did you determine the image coordinates without graphing?

c. Determine the vertex coordinates of image $D''E''F''G''$ if parallelogram *DEFG* is translated 8 units to the right.

d. How did you determine the image coordinates without graphing?

PROBLEM 2 Rotating Geometric Figures on the Coordinate Plane

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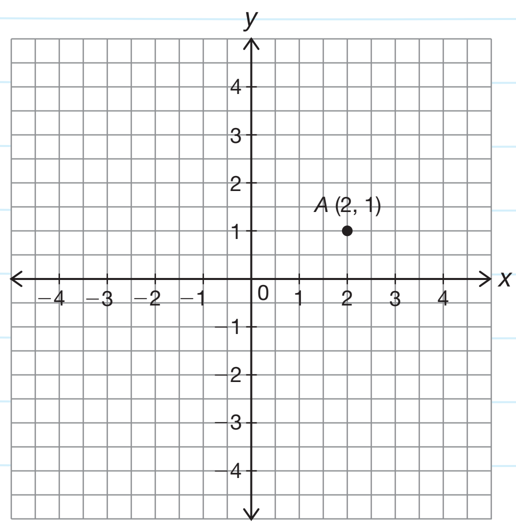
Another transformation that exists in geometry is a *rotation*. A **rotation** is a rigid motion that turns a figure about a fixed point, called the **point of rotation**. The figure is rotated in a given direction for a given angle, called the *angle of rotation*. The **angle of rotation** is the measure of the amount the figure is rotated about the point of rotation. The direction of a rotation can either be clockwise or counterclockwise.

Let's rotate a point about the origin. The origin will be the point of rotation and you will rotate the point 90°, 180°, and 270°.

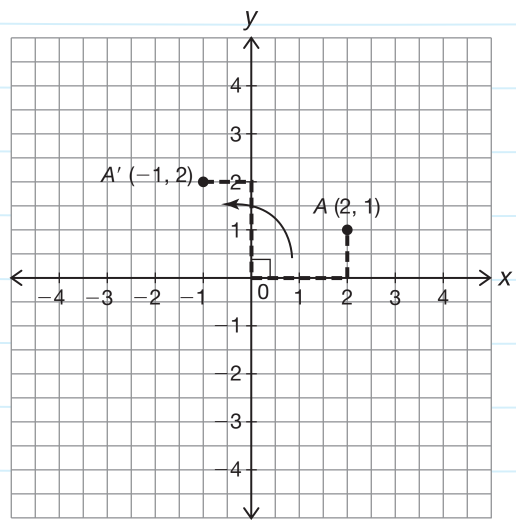
First, let's rotate the point 90° counterclockwise about the origin.

Step 1: Plot a point anywhere in the first quadrant, but *not* at the origin.



Point *A* is plotted at (2, 1).

Step 2: Next, draw a "hook" from the origin to point *A,* using the coordinates and horizontal and vertical line segments as shown.



Step 3: Rotate the "hook" 90° counterclockwise as shown.

Step 4: Point *A*′ is located at (−1, 2). Point *A* has been rotated 90° counterclockwise about the origin.

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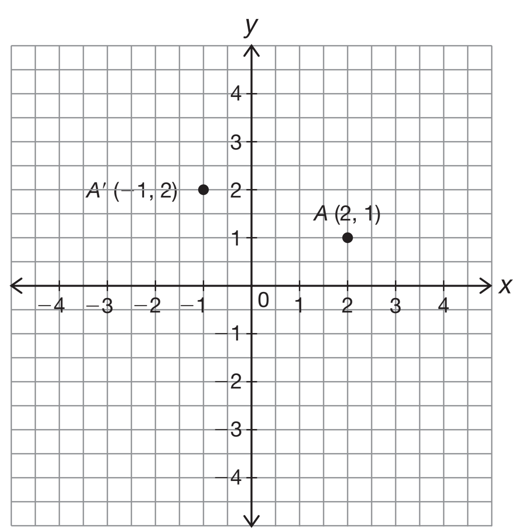
1. What do you notice about the coordinates of point A and the coordinates of point A'?

2. Predict what the coordinates of A'' will be if you rotate A' 90° counterclockwise about the origin.

3. Rotate point A' about the origin 90° counterclockwise on the coordinate plane shown. Label the point A''.



a. What are the coordinates of A''? Was your prediction for the coordinates of A'' correct?

b. What do you notice about the coordinates of A and A''? How are the two points related?

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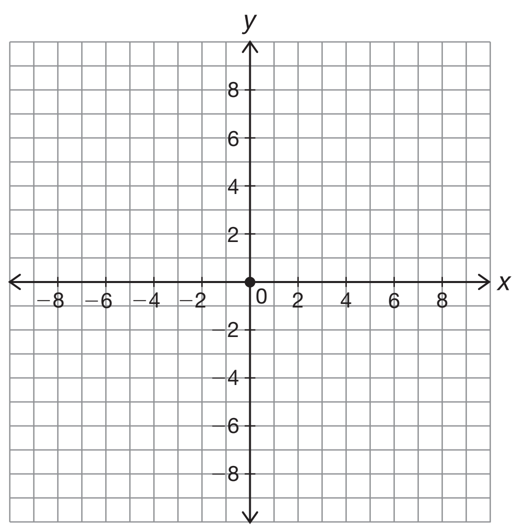
You may have noticed that the values of the *x*- and *y*-coordinates seem to switch places for every 90° rotation about the origin. You may have also noticed that the rotation from *A* to *A″* is a 180° counterclockwise rotation about the origin. In this case, the coordinates of point *A″* are the opposite of the coordinates of point *A*.

You can use the table shown to determine the coordinates of any point after a 90° and 180° counterclockwise rotation about the origin.

Original Point	Coordinates After a Rotation About the Origin 90° Counterclockwise	Coordinates After a Rotation About the Origin 180° Counterclockwise
(x, y)	$(-y, x)$	$(-x, -y)$

Verify that the information in the table is correct by using a test point. You will plot a point on a coordinate plane and rotate it 90° and 180° counterclockwise about the origin.

4. Graph and label point *Q* at (5, 7) on the coordinate plane.



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5. Use the origin (0, 0) as the point of rotation.

a. Rotate Q 90° counterclockwise about the origin. Label the image Q'.

b. Determine the coordinates of Q', then describe how you determined the location of image Q'.

c. Rotate Q 180° counterclockwise about the origin. Label the image Q''.

d. Determine the coordinates of Q'', then describe how you determined the location of image Q''.

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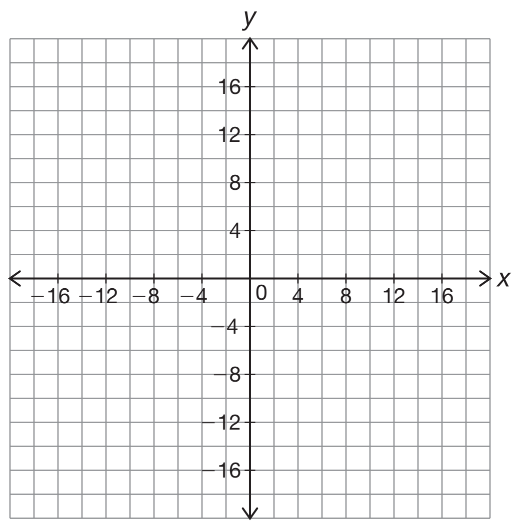
You have been rotating points about the origin on a coordinate plane. However, do you think polygons can also be rotated on the coordinate plane?

You can use models to help show that you *can* rotate polygons on a coordinate plane. However, before we starting modeling the rotation of a polygon on a coordinate plane, let's graph the trapezoid to establish the pre-image.

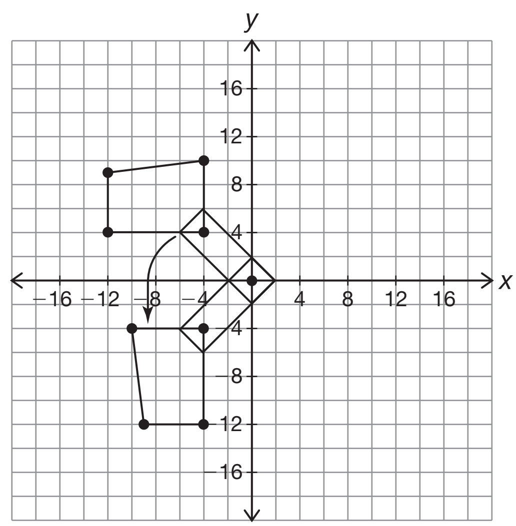
6. Graph trapezoid *ABCD* by plotting the points *A* (−12, 9), *B* (−12, 4), *C* (−4, 4), and *D* (−4, 10).

Now that you have graphed the pre-image, you are ready to model the rotation of the polygon on the coordinate plane.

- First, fold a piece of tape in half and tape it to both sides of the trapezoid you cut out previously.
- Then, take your trapezoid and set it on top of trapezoid *ABCD* on the coordinate plane, making sure that the tape covers the origin (0, 0).
- Finally, put a pin or your pencil point through the tape at the origin and rotate your model counterclockwise. The 90° rotation of trapezoid *ABCD* is shown.



The rotation of trapezoid *ABCD* 90° counterclockwise is shown.



7. Rotate trapezoid *ABCD* about the origin for each given angle of rotation. Graph and label each image on the coordinate plane and record the coordinates in the table.

   a. Rotate trapezoid *ABCD* 90° counterclockwise about the origin to form trapezoid *A′B′C′D′*.

   b. Rotate trapezoid *ABCD* 180° counterclockwise about the origin to form trapezoid *A″B″C″D″*.

Trapezoid *ABCD* (coordinates)	Trapezoid *A′B′C′D′* (coordinates)	Trapezoid *A″B″C″D″* (coordinates)
A (–12, 9)		
B (–12, 4)		
C (–4, 4)		
D (–4, 10)		

8. What similarities do you notice between rotating a single point about the origin and rotating a polygon about the origin?

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Let's consider rotations without graphing.

9. The vertices of parallelogram *DEFG* are *D* (−9, 7), *E* (−12, 2), *F* (−3, 2), and *G* (0, 7).

a. Determine the vertex coordinates of image *D′E′F′G′* if parallelogram *DEFG* is rotated 90° counterclockwise about the origin.

b. How did you determine the image coordinates without graphing?

c. Determine the vertex coordinates of image *D″E″F″G″* if parallelogram *DEFG* is rotated 180° counterclockwise about the origin.

d. How did you determine the image coordinates without graphing?

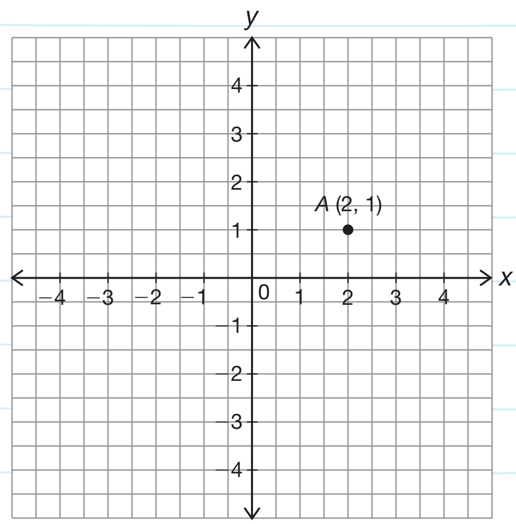
10. Dante claims that if he is trying to determine the coordinates of an image that is rotated 180° about the origin, it does not matter which direction the rotation occurred. Desmond claims that the direction is important to know when determining the image coordinates. Who is correct? Explain why the correct student's rationale is correct.

PROBLEM 3 Reflecting Geometric Figures on the Coordinate Plane

There is a third transformation that can move geometric figures within the coordinate plane. Figures that are mirror images of each other are called *reflections*. A **reflection** is a rigid motion that reflects, or "flips," a figure over a given line called *a line of reflection*. A **line of reflection** is a line over which a figure is reflected so that corresponding points are the same distance from the line.

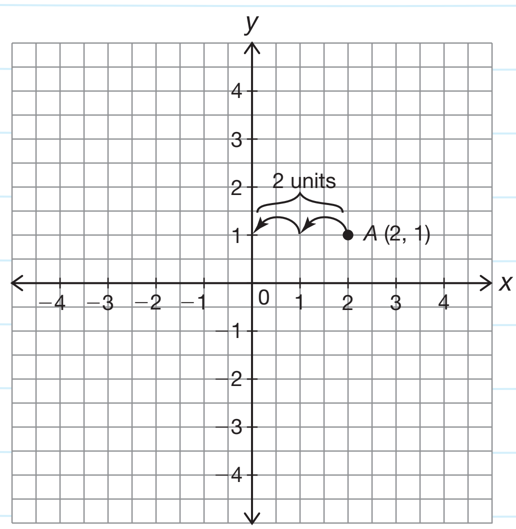
Let's reflect a point over the *y*-axis.

Step 1: Plot a point anywhere in the first quadrant, but *not* at the origin.



Point *A* is plotted at (2, 1).

Step 2: Next, count the number of *x*-units from point *A* to the *y*-axis.



Point *A* is 2 units from the *y*-axis.

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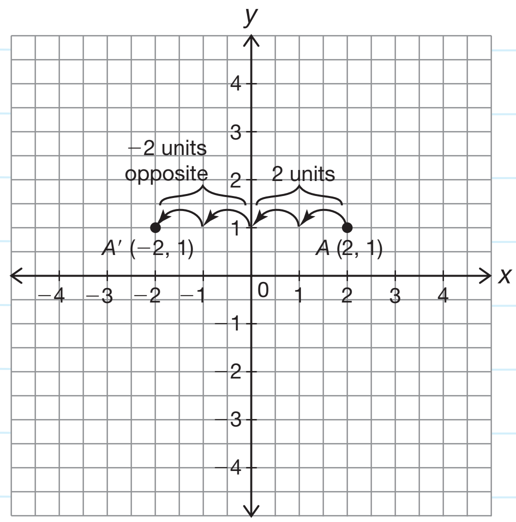
Step 3: Then, count the same number of *x*-units on the opposite side of the *y*-axis to locate the reflection of point *A*. Label the point *A*′.



Step 4: Point *A*′ is located at (−2, 1). Point *A* has been reflected over the *y*-axis.

1. What do you notice about the coordinates of point *A* and the coordinates of image *A*′?

2. Predict the coordinates of *A*″ if point *A* is reflected over the *x*-axis. Explain your reasoning.

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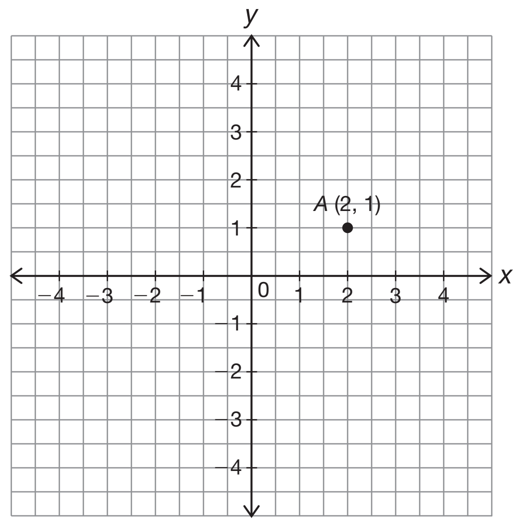
3. Reflect point A over the x-axis on the coordinate plane shown. Verify whether your prediction for the location of the image was correct. Graph the image and label it A''.



4. What do you notice about the coordinates of A and A''?

The coordinates of a pre-image reflected over either the x-axis or the y-axis can be used to determine the coordinates of the image.

You can use the table shown as an efficient way to determine the coordinates of an image reflected over the x- or y-axis.

Original Point	Coordinates of Image After a Reflection Over the x-axis	Coordinates of Image After a Reflection Over the y-axis
(x, y)	$(x, -y)$	$(-x, y)$

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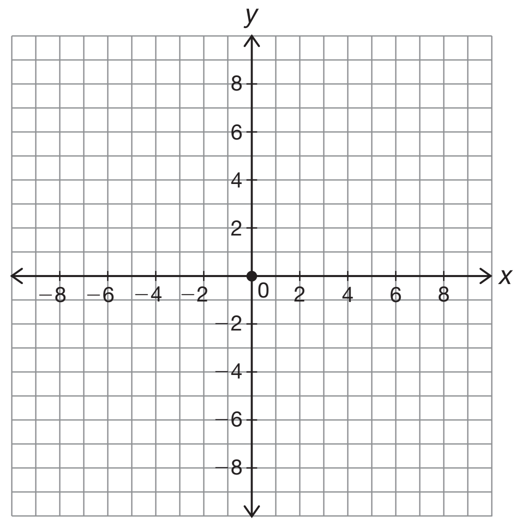
5. Graph point *J* at (5, 7) on the coordinate plane shown.



6. Reflect point *J* over the *y*-axis on the coordinate plane. Label the image *J*′.

7. Determine the coordinates of *J*′. Then describe how you determined the location of image *J*′.

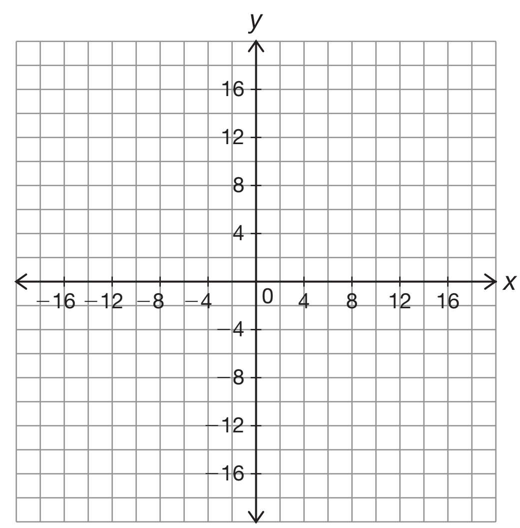
8. Reflect point *J* over the *x*-axis on the coordinate plane. Label the image *J*″.

9. Determine the coordinates of *J*″. Then describe how you determined the location of image *J*″.

You can also reflect polygons on the coordinate plane. You can model the reflection of a polygon across a line of reflection. Just as with rotating a polygon on a coordinate plane, you will first need to establish a pre-image.

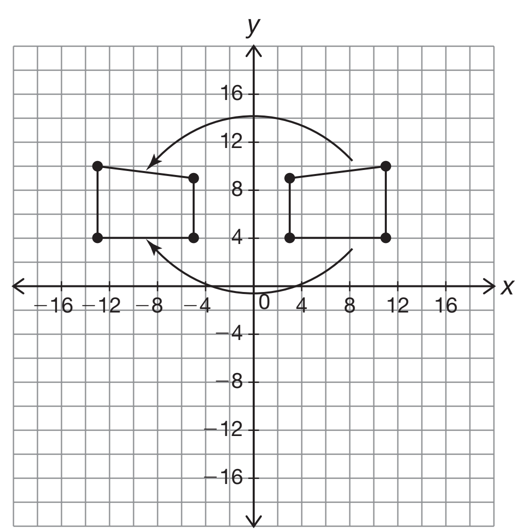
10. Graph trapezoid *ABCD* by plotting the points *A* (3, 9), *B* (3, 4), *C* (11, 4), and *D* (11, 10).

Now that you have graphed the pre-image, you are ready to model the reflection of the polygon on the coordinate plane. For this modeling, you will reflect the polygon over the *y*-axis.



- First, take your trapezoid that you cut out previously and set it on top of trapezoid *ABCD* on the coordinate plane.
- Next, determine the number of units point *A* is from the *y*-axis.
- Then count the same number of units on the opposite side of the *y*-axis to determine where to place the image in Quadrant II.
- Finally, physically flip the trapezoid over the *y*-axis like you are flipping a page in a book.

The reflection of trapezoid *ABCD* over the *y*-axis is shown.



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11. Reflect trapezoid *ABCD* over each given line of reflection. Graph and label each image on the coordinate plane and record each image's coordinates in the table.

a. Reflect trapezoid *ABCD* over the *x*-axis to form trapezoid $A'B'C'D'$.

b. Reflect trapezoid *ABCD* over the *y*-axis to form trapezoid $A''B''C''D''$.

Trapezoid *ABCD* (coordinates)	Trapezoid $A'B'C'D'$ (coordinates)	Trapezoid $A''B''C''D''$ (coordinates)
A **(3, 9)**		
B **(3, 4)**		
C **(11, 4)**		
D **(11, 10)**		

12. What similarities do you notice between reflecting a single point over the *x*- or *y*-axis and reflecting a polygon over the *x*- or *y*-axis?

Let's consider reflections without graphing.

13. The vertices of parallelogram *DEFG* are *D* (−9, 7), *E* (−12, 2), *F* (−3, 2), and *G* (0, 7).

a. Determine the vertex coordinates of image $D'E'F'G'$ if parallelogram *DEFG* is reflected over the *x*-axis.

b. How did you determine the image coordinates without graphing?

c. Determine the vertex coordinates of image *D″E″F″G″* if parallelogram *DEFG* is reflected over the *y*-axis.

d. How did you determine the image coordinates without graphing?

Talk the Talk

1. The vertices of rectangle *PQRS* are *P* (6, 8), *Q* (6, 2), *R* (−3, 2), and *S* (−3, 8). Describe the translation used to form each rectangle given each image's coordinates. Explain your reasoning.

a. *P′* (1, 8), *Q′* (1, 2), *R′* (−8, 2), and *S′* (−8, 8)

b. *P″* (6, 14.5), *Q″* (6, 8.5), *R″* (−3, 8.5), and *S″* (−3, 14.5)

2. The vertices of rectangle *JKLM* are *J* (6, 8), *K* (6, 2), *L* (−3, 2), and *M* (−3, 8). Describe the rotation used to form each rectangle. Explain your reasoning.

a. *J′* (−8, 6), *K′* (−2, 6), *L′* (−2, −3), and *M′* (−8, −3)

b. *J″* (−6, −8), *K″* (−6, −2), *L″* (3, −2), and *M″* (3, −8)

3. The vertices of rectangle *NOPQ* are *N* (8, 8), *O* (8, 2), *P* (−3, 2), and *Q* (−3, 8). Describe the reflection used to form each rectangle. Explain your reasoning.

a. N' (−8, 8), O' (−8, 2), P' (3, 2), and Q' (3, 8)

b. N'' (8, −8), O'' (8, −2), P'' (−3, −2), and Q'' (−3, −8)

4. Complete this sentence:

Images that result from a translation, rotation, or reflection are (always, sometimes, or never) congruent to the original figure.

Be prepared to share your solutions and methods.

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13.2

All the Same to You

Congruent Triangles

LEARNING GOALS

In this lesson, you will:

- Identify corresponding sides and corresponding angles of congruent triangles.
- Explore the relationship between corresponding sides of congruent triangles.
- Explore the relationship between corresponding angles of congruent triangles.
- Write statements of triangle congruence.
- Identify and use rigid motion to create new images.

KEY TERMS

- congruent angles
- corresponding sides
- corresponding angles

In mathematics, when a geometric figure is transformed, the size and shape of the figure do not change. However, in physics, things are a little different. An idea known as length contraction explains that when an object is in motion, its length appears to be slightly less than it really is. This cannot be seen with everyday objects because they do not move fast enough. To truly see this phenomenon you would have to view an object moving close to the speed of light. In fact, if an object was moving past you at the speed of light, the length of the object would seem to be practically zero!

This theory is very difficult to prove and yet scientists came up with the idea in the late 1800s. How do you think scientists test and prove length contraction? Do you think geometry is used in these verifications?

PROBLEM 1 Understanding Congruence

In a previous lesson, you determined that when you translate, rotate, or reflect a figure, the resulting image is the same size and the same shape as the pre-image. Therefore, the image and the pre-image are said to be congruent.

Recall that congruent line segments are line segments that have the same length. Congruent triangles are triangles that are the same size and the same shape.

If the length of line segment *AB* is equal to the length of line segment *DE*, the relationship can be expressed using symbols.

- $AB = DE$ is read as "the distance between *A* and *B* is equal to the distance between *D* and *E*."
- $m\overline{AB} = m\overline{DE}$ is read as "the measure of line segment *AB* is equal to the measure of line segment *DE*."
- $\overline{AB} \cong \overline{DE}$ is read as "line segment *AB* is congruent to line segment *DE*."

Congruent angles are angles that are equal in measure.

If the measure of angle *A* is equal to the measure of angle *D*, the relationship can be expressed using symbols.

- $m\angle A = m\angle D$ is read as "the measure of angle *A* is equal to the measure of angle *D*."
- $\angle A \cong \angle D$ is read as "angle *A* is congruent to angle *D*."

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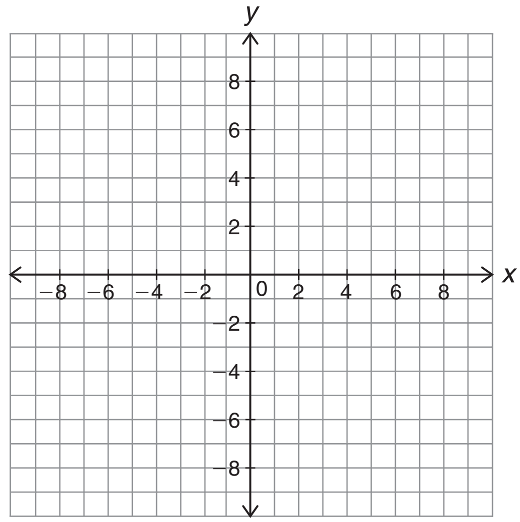
Let's explore the properties of congruent triangles.

1. Graph triangle *ABC* by plotting the points *A* (8, 10), *B* (1, 2), and *C* (8, 2).



a. Classify triangle *ABC*. Explain your reasoning.

b. Use the Pythagorean Theorem to determine the length of side *AB*.

2. Translate triangle *ABC* 10 units to the left to form triangle *DEF*. Graph triangle *DEF* and list the coordinates of points *D, E,* and *F*.

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Corresponding sides are sides that have the same relative positions in corresponding geometric figures.

Triangle *ABC* and triangle *DEF* in Question 1 are the same size and the same shape. Each side of triangle *ABC* matches, or corresponds to, a specific side of triangle *DEF*.

3. Given what you know about corresponding sides of congruent triangles, predict the side lengths of triangle *DEF*.

4. Verify your prediction.

a. Identify the pairs of corresponding sides of triangle *ABC* and triangle *DEF*.

b. Determine the side lengths of triangle *DEF*.

c. Compare the lengths of the sides of triangle *ABC* to the lengths of the corresponding sides of triangle *DEF*. What do you notice?

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5. In general, what can you conclude about the relationship between the corresponding sides of congruent triangles?

Use triangle *ABC* and triangle *DEF* from Question 1 to answer each question.

6. Use a protractor to determine the measures of ∠*A*, ∠*B*, and ∠*C*.

Each angle in triangle *ABC* corresponds to a specific angle in triangle *DEF*. **Corresponding angles** are angles that have the same relative positions in corresponding geometric figures.

7. What would you predict to be true about the measures of corresponding angles of congruent triangles?

8. Verify your prediction.

a. Identify the corresponding angles of triangle *ABC* and triangle *DEF*.

b. Use a protractor to determine the measures of angles *D*, *E*, and *F*.

c. Compare the measures of the angles of triangle *ABC* to the measures of the corresponding angles of triangle *DEF*.

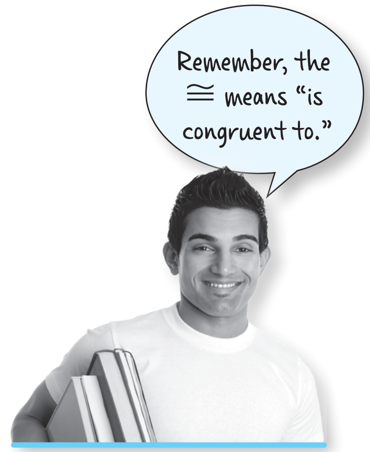
9. In general, what can you conclude about the relationship between the corresponding angles of congruent triangles?

PROBLEM 2 Statements of Triangle Congruence

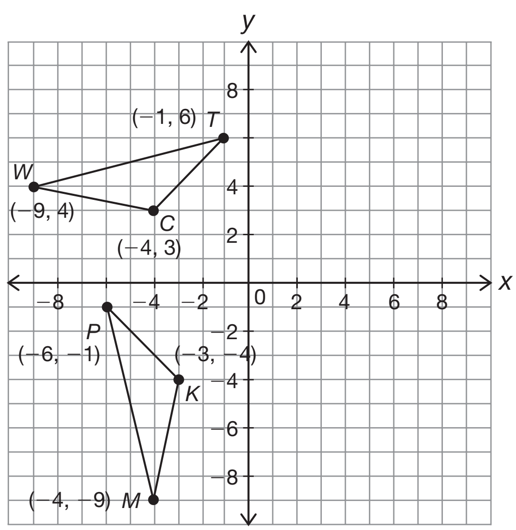
1. Consider the congruence statement $\triangle JRB \cong \triangle MNS$.

   a. Identify the congruent angles.

   b. Identify the congruent sides.



2. Analyze the two triangles shown.



   a. Determine the transformation used to create triangle *PMK*.

   b. Does the transformation preserve the size and shape of the triangle in this problem situation? Why or why not?

   c. Write a triangle congruence statement for the triangles.

   d. Identify the congruent angles.

   e. Identify the congruent sides.

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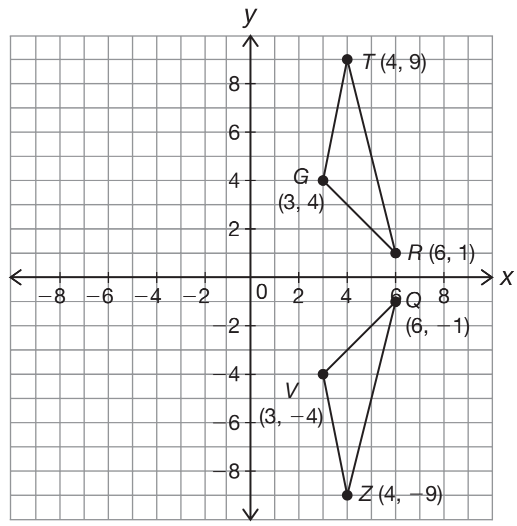
3. Analyze the two triangles shown.



a. Determine the transformation used to create triangle *ZQV*.

b. Does the transformation preserve the size and shape of the triangle in this problem situation? Why or why not?

c. Write a triangle congruence statement for the triangles shown.

d. Identify the congruent angles.

e. Identify the congruent sides.

Talk the Talk

1. Given any triangle on a coordinate plane, how can you create a different triangle that you know will be congruent to the original triangle?

2. Describe the properties of congruent triangles.

Be prepared to share your solutions and methods.

13.3

Side-Side-Side

SSS Congruence Theorem

LEARNING GOALS

In this lesson, you will:

- Explore the Side-Side-Side Congruence Theorem through constructions.
- Explore the Side-Side-Side Congruence Theorem on the coordinate plane.

KEY TERMS

- theorem
- postulate
- Side-Side-Side Congruence Theorem

Have you ever tried to construct something from scratch—a model car or a bird house, for example? If you have, you have probably discovered that it is a lot more difficult than it looks. To build something accurately, you must have a plan in place. You must think about materials you will need, measurements you will make, and the amount of time it will take to complete the project. You may need to make a model or blueprint of what you are building. Then, when the actual building begins, you must be very precise in all your measurements and cuts. The difference of half an inch may not seem like much, but it could mean the wall of your bird house is too small and now you may have to start again!

You will be constructing triangles throughout the next 4 lessons. While you won't be cutting or building anything, it is still important to measure accurately and be precise. Otherwise, you may think your triangles are accurate even though they're not!

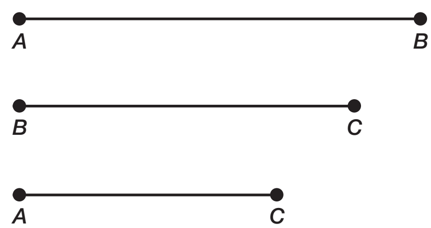
PROBLEM 1 Can You Build It?

In mathematics you often have to prove a solution is correct. In geometry, *theorems* are used to verify statements. A **theorem** is a statement that can be proven true using definitions, *postulates*, or other theorems. A **postulate** is a mathematical statement that cannot be proved but is considered true.

While you can assume that all duplicated or transformed triangles are congruent, mathematically, you need to use a theorem to prove it.

The *Side-Side-Side Congruence Theorem* is one theorem that can be used to prove triangle congruence. The **Side-Side-Side Congruence Theorem** states that if three sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent.

1. Use the given line segments to construct triangle *ABC*. Then, write the steps you performed to construct the triangle.



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2. Analyze the triangle you created.

   a. Classify $\triangle ABC$. Explain your reasoning.

b. Compare your triangle to your classmates' triangles. Are the triangles congruent? Why or why not?

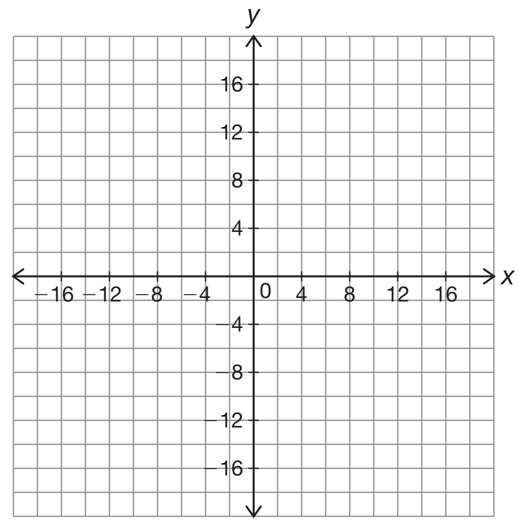
c. How many different triangles can be formed given the lengths of three distinct sides?

3. Rico compares his triangle with his classmate Annette's. Rico gets out his ruler and protractor to verify the triangles are congruent. Annette states he does not need to do that. Who is correct? Explain your reasoning.

PROBLEM 2 Get Back on the Plane

In the previous problem, you proved that two triangles are congruent if three sides of one triangle are congruent to the corresponding sides of another triangle. When dealing with triangles on the coordinate plane, measurement must be used to prove congruence.

1. Graph triangle *ABC* by plotting the points *A* (8, −5), *B* (4, −12), and *C* (12, −8).



2. How can you determine the length of each side of this triangle?

3. Calculate the length of each side of triangle *ABC*. Record the measurements in the table.

Side of Triangle *ABC*	Length of Side
$\overline{AB}$	
$\overline{BC}$	
$\overline{AC}$	

4. Translate line segments AB, BC, and AC up 7 units to form triangle $A'B'C'$. Graph the image.

5. Calculate the length of each side of triangle $A'B'C'$. Record the measurements in the table.

Side of Triangle $A'B'C'$	Length of Side
$\overline{A'B'}$	
$\overline{B'C'}$	
$\overline{A'C'}$	

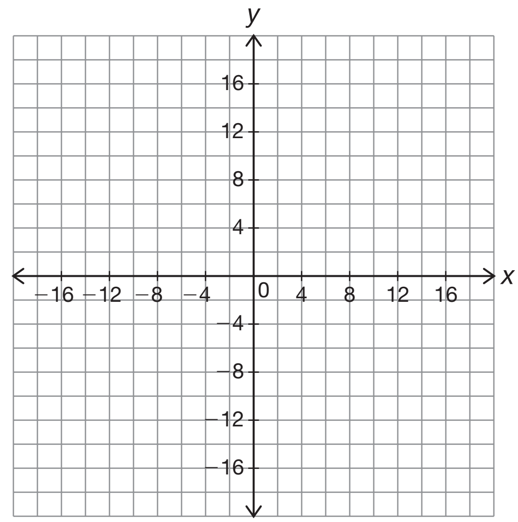
6. Are the corresponding sides of the pre-image and image congruent? Explain your reasoning.

13

7. Do you need to determine the measures of the angles to verify that the triangles are congruent? Explain why or why not.

PROBLEM 3 Flipping for Congruence

1. Graph triangle *ABC* using the same coordinates as in Problem 1.



2. Reflect line segments *AB*, *BC*, and *AC* over the *x*-axis to form triangle *A″B″C″*.

3. Calculate the length of each side of triangle *A′B′C′*. Record the measurements in the table.

Side of Triangle *D′E′F′*	Length of Side
$\overline{A'B'}$	
$\overline{B'C'}$	
$\overline{A'C'}$	

13

4. Are the triangles congruent? Explain your reasoning.

Be prepared to share your solutions and methods.

13

Side-Angle-Side

SAS Congruence Theorem

13.4

OBJECTIVES

In this lesson, you will:

- Explore Side-Angle-Side Congruence Theorem using constructions.
- Explore Side-Angle-Side Congruence Theorem on the coordinate plane.

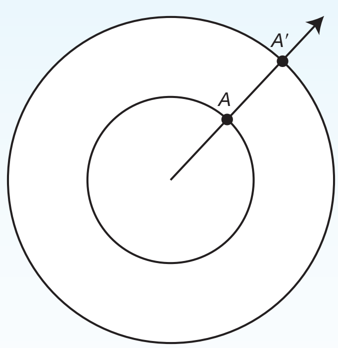
KEY TERMS

- Side-Angle-Side Congruence Theorem
- included angle

The smaller circle you see here has an infinite number of points. And the larger circle has an infinite number of points. But since the larger circle is, well, larger, shouldn't it have more points than the smaller circle?

Mathematicians use one-to-one correspondence to determine if two sets are equal. If you can show that each object in a set corresponds to one and only one object in another set, then the two sets are equal.

Look at the circles. Any ray drawn from the center will touch only two points—one on the smaller circle and one on the larger circle. This means that both circles contain the same number of points! Can you see how correspondence was used to come up with this answer?



PROBLEM 1 Using Constructions to Support Side-Angle-Side

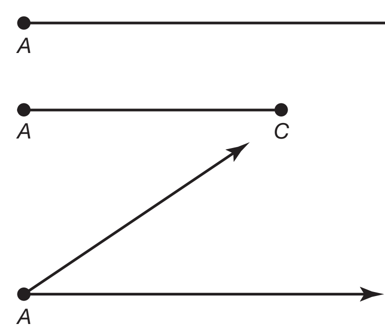
So far in this chapter, you have determined the congruence of two triangles by proving that if the sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent.

There is another way to determine if two triangles are congruent that does not involve knowledge of three sides. You will prove the *Side-Angle-Side Congruence Theorem*.

The **Side-Angle-Side Congruence Theorem** states that if two sides and the *included angle* of one triangle are congruent to the corresponding sides and the included angle of the second triangle, then the triangles are congruent. An **included angle** is the angle formed by two sides of a triangle.

First, let's prove this theorem through construction.

1. Construct $\triangle ABC$ using the two line segments and included angle shown. Then, write the steps you performed to construct the triangle.



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2. How does the length of side *BC* compare to the length of your classmates' side *BC*?

3. Use a protractor to measure angle *B* and angle *C* in triangle *ABC*.

4. How do the measures of your corresponding angles compare to the measures of your classmates' corresponding angles?

5. Is your triangle congruent to your classmates' triangles? Why or why not?

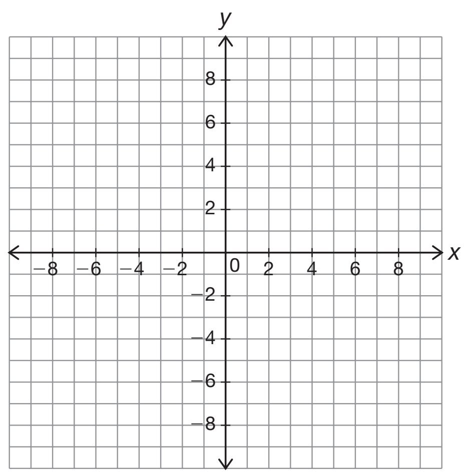
6. If you were given one of the non-included angles, $\angle C$ or $\angle B$, instead of $\angle A$, do you think everyone in your class would have constructed an identical triangle? Explain your reasoning.

PROBLEM 2 Using Rotation to Support Side-Angle-Side

Through your construction, you and your classmates constructed congruent triangles using two given sides and the included angle of a triangle.

Let's now try to prove the Side-Angle-Side Theorem on the coordinate plane using algebra.

1. Graph triangle *ABC* by plotting the points *A* (5, 9), *B* (2, 3), and *C* (7, 2).



2. Calculate the length of each side of triangle *ABC* and record the measurements in the table. Record exact measurements.

Side of Triangle *ABC*	Length of Side
$\overline{AB}$	
$\overline{BC}$	
$\overline{AC}$	

13

3. Rotate side *AB*, side *BC*, and included angle *B*, in triangle *ABC* 270° counterclockwise. Then connect points *A*′ and *C*′ to form triangle *A*′*B*′*C*′. Use the table to record the image coordinates.

Triangle *ABC* (coordinates)	Triangle *A*′*B*′*C*′ (coordinates)
A (5, 9)	
B (2, 3)	
C (14, 3)	

4. Calculate the length of each side of triangle *A*′*B*′*C*′ and record the measurements in the table. Record exact measurements.

Side of Triangle *A*′*B*′*C*′	Length of Side
$\overline{A'B'}$	
$\overline{B'C'}$	
$\overline{A'C'}$	

13

5. What do you notice about the corresponding side lengths of the pre-image and the image?

6. Use a protractor to measure angle *B* of triangle *ABC* and angle *B′* of triangle *A′B′C′*.

   a. What are the measures of each angle?

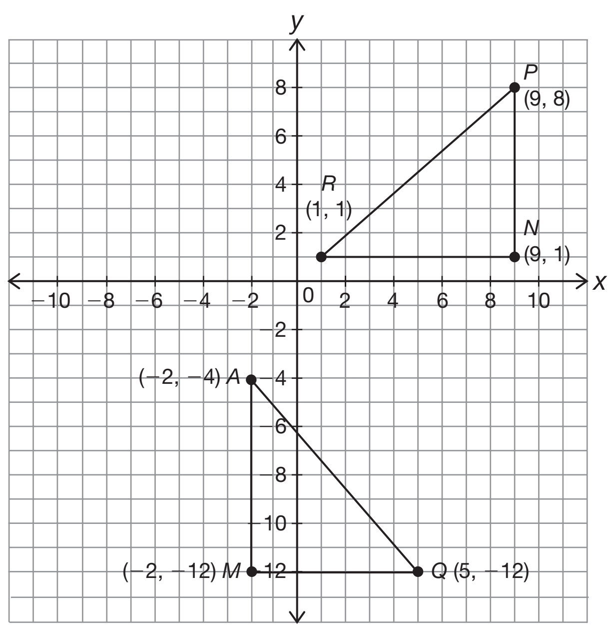
   b. What does this information tell you about the corresponding angles of the two triangles?

You have shown that the corresponding sides of the image and pre-image are congruent. Therefore, the triangles are congruent by the SSS Congruence Theorem.

You have also used a protractor to verify that the corresponding included angles of each triangle are congruent.

In conclusion, when two side lengths of one triangle and the measure of the included angle are equal to the two corresponding side lengths and the measure of the included angle of another triangle, the two triangles are congruent by the SAS Congruence Theorem.

7. Use the SAS Congruence Theorem and a protractor to determine if the two triangles drawn on the coordinate plane shown are congruent. Use a protractor to determine the measures of the included angles.



PROBLEM 3 Representing Congruence

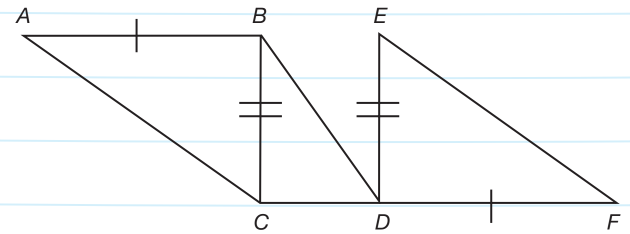
Congruent line segments and congruent angles are often denoted using special markers, rather than given measurements.

Slash markers can be used to indicate congruent line segments. When multiple line segments contain a single slash marker, this implies that all of those line segments are congruent. Double and triple slash markers can also be used to denote other line segment congruencies.

Arc markers can be used to indicate congruent angles. When multiple angles contain a single arc marker, this implies that those angles are congruent. Double and triple arc markers can also be used to denote other angle congruencies.

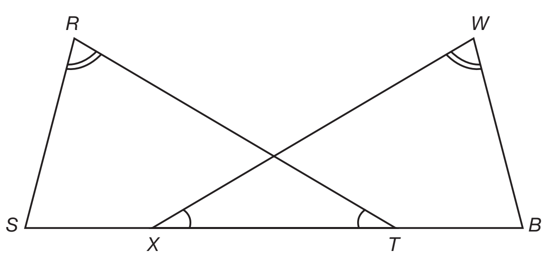
The markers on the diagram indicate congruent line segments.

$\overline{AB} \cong \overline{DF}$ and $\overline{BC} \cong \overline{ED}$

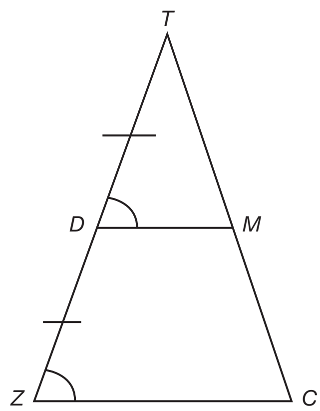


1. Write the congruence statements represented by the markers in each diagram.

a.



b.



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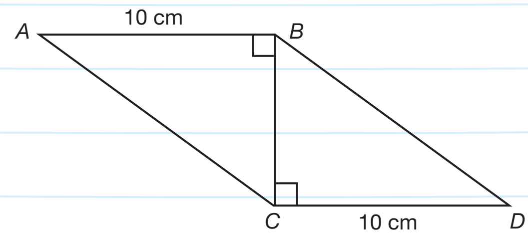
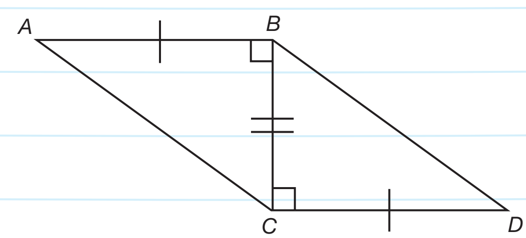
You can analyze diagrams and use SAS and SSS to determine if triangles are congruent.

Analyze the figure shown to determine if $\triangle ABC$ is congruent to $\triangle DCB$.



Notice, $m\overline{AB} = 10$ cm and $m\overline{DC} = 10$ cm and they are corresponding sides of the two triangles. Also notice that $\angle ABC$ and $\angle DCB$ are right angles, and they are corresponding angles of the two triangles.

In order to prove that the two triangles are congruent using SAS, you need to show that another side of triangle *ABC* is congruent to another side of triangle *DCB*. Notice that the two triangles share a side. Because line segment *BC* is the same as line segment *CB*, you know that these two line segments are congruent.



So, $\triangle ABC \cong \triangle DCB$ by the SAS Congruence Theorem.

2. Write the three congruence statements that show $\triangle ABC \cong \triangle DCB$ by the SAS Congruence Theorem.

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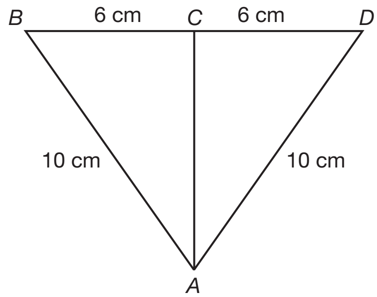
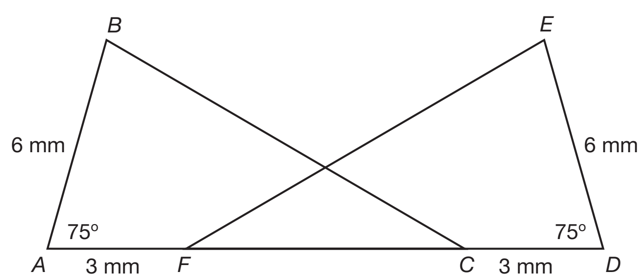
3. Determine if there is enough information to prove that the two triangles are congruent by SSS or SAS. Write the congruence statements to justify your reasoning.

a. $\triangle ABC \overset{?}{\cong} \triangle ADC$



b. $\triangle ABC \overset{?}{\cong} \triangle DEF$



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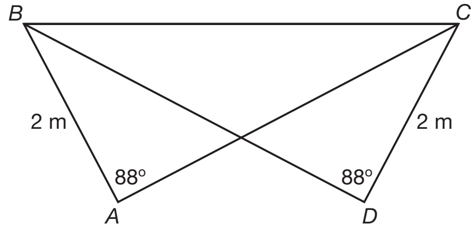
4. Simone says that since the two triangles shown have two pairs of congruent corresponding sides and congruent corresponding angles, then the triangles are congruent by SAS. Is Simone correct? Explain your reasoning.



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Be prepared to share your solutions and methods.

You Shouldn't Make Assumptions

13.5

Angle-Side-Angle Congruence Theorem

LEARNING GOALS

In this lesson, you will:

- Explore the Angle-Side-Angle Congruence Theorem using constructions.
- Explore the Angle-Side-Angle Congruence Theorem on the coordinate plane.

KEY TERMS

- Angle-Side-Angle Congruence Theorem
- included side

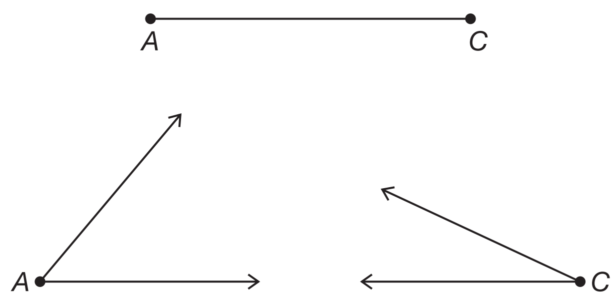
"Don't judge a book by its cover." What does this saying mean to you? Usually it is said to remind someone not to make assumptions. Just because something (or someone!) looks a certain way on the outside, until you really get into it, you don't know the whole story. Often in geometry it is easy to make assumptions. You assume that two figures are congruent because they look congruent. You assume two lines are perpendicular because they look perpendicular. Unfortunately, mathematics and assumptions do not go well together. Just as you should not judge a book by its cover, you should not assume anything about a measurement just because it looks a certain way.

Have you made any geometric assumptions so far in this chapter? Was your assumption correct or incorrect? Hopefully it will only take you one incorrect assumption to learn to not assume!

PROBLEM 1 Putting the Pieces Together

So far you have looked at the Side-Side-Side and Side-Angle-Side Congruence Theorems. But are there other theorems that prove triangle congruence as well?

1. Use the given two angles and line segment to construct triangle *ABC*. Then, write the steps your performed to construct the triangle.



2. Compare your triangle to your classmates' triangles. Are the triangles congruent? Why or why not?

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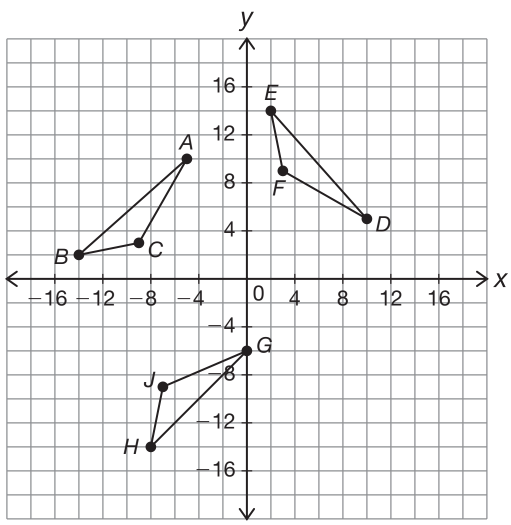
3. Wendy says that if the line segment and angles had not been labeled, then all the triangles would not have been congruent. Ian disagrees and says that there is only one way to put two angles and a side together to form a triangle whether they are labeled or not. Who is correct? Explain your reasoning.

You just used construction to prove the *Angle-Side-Angle Congruence Theorem*. The **Angle-Side-Angle Congruence Theorem** states that if two angles and the *included side* of one triangle are congruent to the corresponding two angles and the included side of another triangle, then the triangles are congruent. An **included side** is the side between two angles of a triangle.

PROBLEM 2 How Did You Get There?

1. Analyze triangles *ABC* and *DEF*.



a. Describe the possible transformation(s) that could have occurred to transform pre-image *ABC* into image *DEF*.

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b. Identify two pairs of corresponding angles and a pair of corresponding included sides that could be used to determine congruence through the ASA Congruence Theorem.

c. Use the ASA Congruence Theorem and a protractor to determine if the two triangles are congruent.

2. Analyze triangles *DEF* and *GHJ*.

a. Describe the possible transformation(s) that could have occurred to transform pre-image *DEF* to image *GHJ*.

b. Identify two pairs of corresponding angles and a pair of corresponding included sides that could be used to determine congruence through the ASA Congruence Theorem.

c. Use the ASA Congruence Theorem and a protractor to determine if the two triangles are congruent.

3. Based on your solution to Question 2, part (c), what can you conclude about the relationship between triangle *ABC* and triangle *GHJ*? Explain your reasoning.

Be prepared to share your solutions and methods.

13

13.6

Ahhhhh…We're Sorry We Didn't Include You!

Angle-Angle-Side Congruence Theorem

OBJECTIVES

In this lesson, you will:

- Explore Angle-Angle-Side Congruence Theorem using constructions.
- Explore Angle-Angle-Side Congruence Theorem on the coordinate plane.

KEY TERMS

- Angle-Angle-Side Congruence Theorem
- non-included side

Sometimes, good things must come to an end, and that can be said for determining if triangles are congruent given certain information.

You have used many different theorems to prove that two triangles are congruent based on different criteria. Specifically,

- Side-Side-Side Congruence Theorem
- Side-Angle Side Congruence Theorem
- and Angle-Side-Angle Congruence Theorem.

So, do you think there are any other theorems that must be used to prove that two triangles are congruent? Here's a hint: we have another lesson—so there must be at least one more congruence theorem!

PROBLEM 1 Using Constructions to Support AAS

There is another way to determine if two triangles are congruent that is different from the congruence theorems you have already proven. You will prove the *Angle-Angle-Side Congruence Theorem*.

The **Angle-Angle-Side Congruence Theorem** states that if two angles and a *non-included side* of one triangle are congruent to the corresponding angles and the corresponding non-included side of a second triangle, then the triangles are congruent. The **non-included side** is a side that is *not* located between the two angles.

First, you will prove this theorem through construction.

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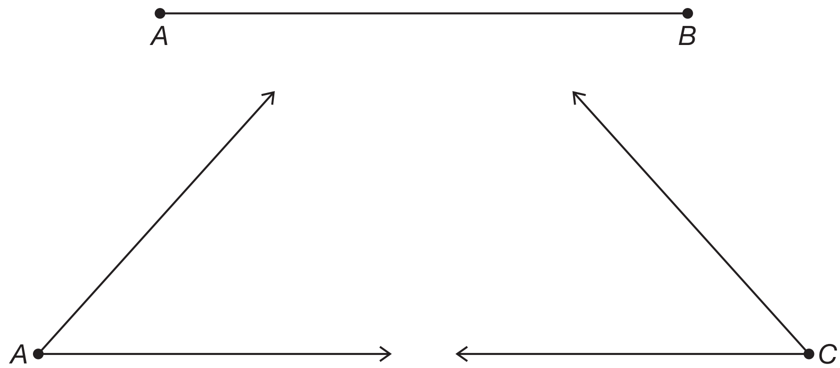
1. Construct triangle *ABC* given line segment *AB* and angles *A* and *C*. Then, write the steps you performed to construct the triangle.



2. How does the length of side *AB* compare to the length of your classmates' side *AB*?

3. Use a protractor to measure angle *A* and angle *C* in triangle *ABC*. What do you notice about your angle measures and your classmates' angle measures?

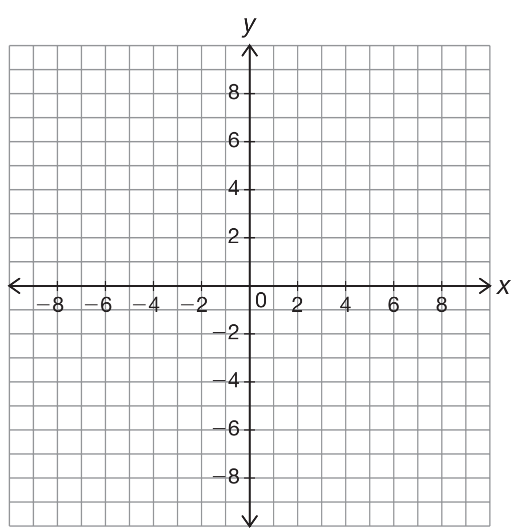
4. Thomas claims that his constructed triangle is not congruent because he drew a vertical starter line that created a triangle that has side *AB* being vertical rather than horizontal. Denise claims that all the constructed triangles are congruent even though Thomas's triangle looks different. Who's correct? Why is this student correct?

5. Is your triangle congruent to your classmates' triangles? Why or why not?

PROBLEM 2 Using Reflection to Support AAS

If two angles and the non-included side of a triangle are reflected is the image of the triangle congruent to the pre-image of the triangle?

1. Graph triangle *ABC* by plotting the points *A* (−3, −6), *B* (−9, −10), and *C* (−1, −10).



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2. Use the Distance Formula to calculate the length of each side of triangle *ABC*. Record the exact measurements in the table.

Side of Triangle *ABC*	Length of Side
$\overline{AB}$	
$\overline{BC}$	
$\overline{AC}$	

3. Reflect angle A, angle B, and side BC over the line of reflection $y = -2$ to form angle D, angle E, and side EF. Then connect points D and E to form triangle DEF. Record the image coordinates in the table.

Triangle *ABC* (coordinates)	Triangle *DEF* (coordinates)
A (−3, −6)	
B (−9, −10)	
C (−1, −10)	

4. Use the Distance Formula to calculate the length of each side of triangle DEF. Record the exact measurements in the table.

Side of Triangle *DEF*	Length of Side
DE	
EF	
DF	

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5. Compare the corresponding side lengths of the pre-image and image. What do you notice?

You have shown that the corresponding sides of the image and pre-image are congruent. Therefore, the triangles are congruent by the SSS Congruence Theorem. However, you are proving the Angle-Angle-Side Congruence Theorem. Therefore, you need to verify if angle *A* and angle *C* are congruent to the corresponding angles in triangle *DEF*.

6. Use a protractor to determine the angle measures of each triangle.

a. What is the measure of angle *A* and angle *C*?

b. Which angles in triangle *DEF* correspond to angle *A* and angle *C*?

c. What do you notice about the measures of the corresponding angles in the triangles? What can you conclude from this information?

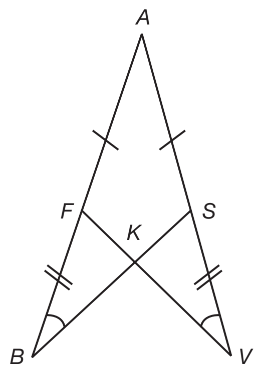
You have used a protractor to verify that the corresponding angles of the two triangles are congruent.

In conclusion, when the measure of two angles and the length of the non-included side of one triangle are equal to the measure of the two corresponding angles and the length of the non-included side of another triangle, the two triangles are congruent by the AAS Congruence Theorem.

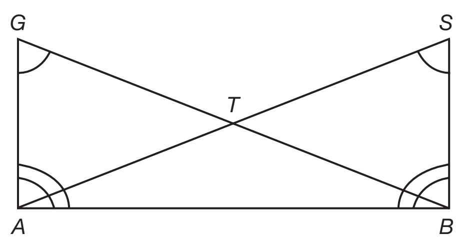
PROBLEM 3 ASA Congruence or AAS Congruence

Determine if there is enough information to prove that the two triangles are congruent by ASA or AAS. Write the congruence statements to justify your reasoning.

1. $\triangle ABS \overset{?}{\cong} \triangle AVF$



2. $\triangle GAB \overset{?}{\cong} \triangle SBA$



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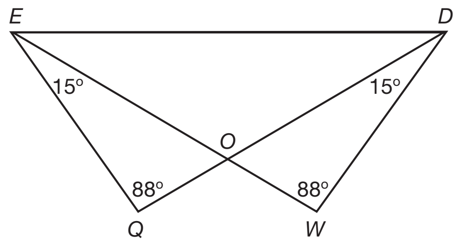
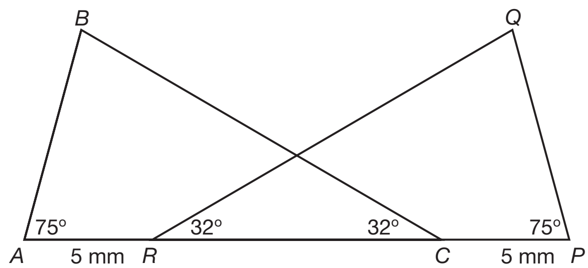
3. $\triangle EQD \overset{?}{\cong} \triangle DWE$

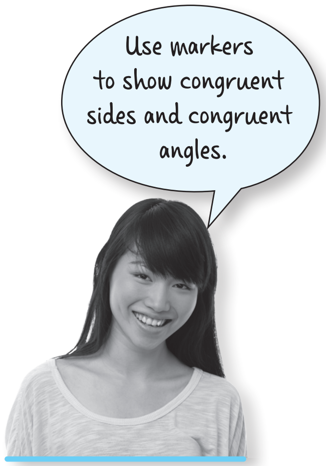


4. $\triangle ABC \overset{?}{\cong} \triangle PQR$



Talk the Talk

This chapter focused on four methods that you can use to prove that two triangles are congruent. Complete the graphic organizer by providing an illustration of each theorem.



Be prepared to share your solutions and methods.

Side-Side-Side Congruence Theorem

Side-Angle-Side Congruence Theorem

Triangle Congruence Theorems

Angle-Side-Angle Congruence Theorem

Angle-Angle-Side Congruence Theorem

Chapter 13 Summary

KEY TERMS

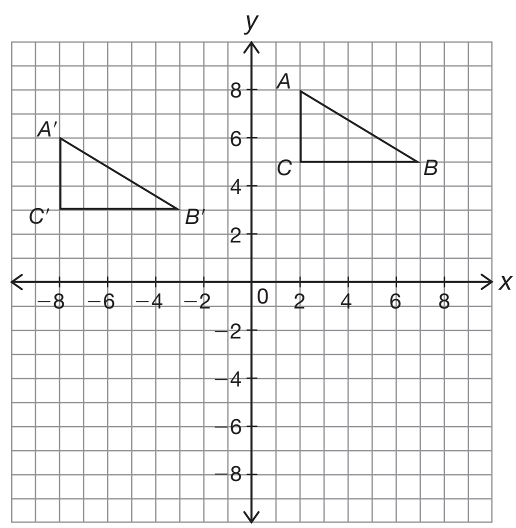
- rotation (13.1)
- point of rotation (13.1)
- angle of rotation (13.1)
- reflection (13.1)
- line of reflection (13.1)
- congruent angles (13.2)
- corresponding sides (13.2)
- corresponding angles (13.2)
- theorem (13.3)
- postulate (13.3)
- Side-Side-Side Congruence Theorem (13.3)
- Side-Angle-Side Congruence Theorem (13.4)
- included angle (13.4)
- Angle-Side-Angle Congruence Theorem (13.5)
- included side (13.5)
- Angle-Angle-Side Congruence Theorem (13.6)
- non-included side (13.6)

13.1 Translating Triangles on the Coordinate Plane

A translation is a rigid motion that slides each point of a figure the same distance and direction.

Example

Triangle *ABC* has been translated 10 units to the left and 2 units down to create triangle *A′B′C′*.



The coordinates of triangle *ABC* are *A* (2, 8), *B* (7, 5), and *C* (2, 5).

The coordinates of triangle *A′B′C′* are *A′* (−8, 6), *B′* (−3, 3), and *C′* (−8, 3).

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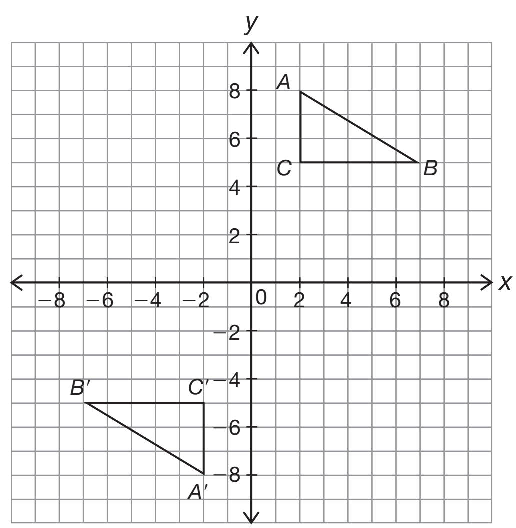
13.1 Rotating Triangles in the Coordinate Plane

A rotation is a rigid motion that turns a figure about a fixed point, called the point of rotation. The figure is rotated in a given direction for a given angle, called the angle of rotation. The angle of rotation is the measure of the amount the figure is rotated about the point of rotation. The direction of a rotation can either be clockwise or counterclockwise. To determine the new coordinates of a point after a 90° counterclockwise rotation, change the sign of the *y*-coordinate of the original point and then switch the *x*-coordinate and the *y*-coordinate. To determine the new coordinates of a point after a 180° rotation, change the signs of the *x*-coordinate and the *y*-coordinate of the original point.

Example

Triangle *ABC* has been rotated 180° counterclockwise about the origin to create triangle *A′B′C′*.



The coordinates of triangle *ABC* are *A* (2, 8), *B* (7, 5), and *C* (2, 5).

The coordinates of triangle *A′B′C′* are *A′* (−2, −8), *B′* (−7, −5), and *C′* (−2, −5).

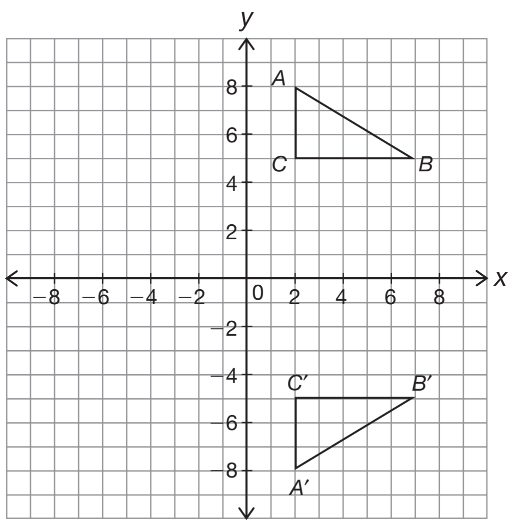
13

13.1 Reflecting Triangles on a Coordinate Plane

A reflection is a rigid motion that reflects or "flips" a figure over a given line called a line of reflection. Each point in the new triangle will be the same distance from the line of reflection as the corresponding point in the original triangle. To determine the coordinates of a point after a reflection across the *x*-axis, change the sign of the *y*-coordinate of the original point. The *x*-coordinate remains the same. To determine the coordinates of a point after a reflection across the *y*-axis, change the sign of the *x*-coordinate of the original point. The *y*-coordinate remains the same.

Example

Triangle *ABC* has been reflected across the *x*-axis to create triangle *A′B′C′*.



The coordinates of triangle *ABC* are *A* (2, 8), *B* (7, 5), and *C* (2, 5).

The coordinates of triangle *A′B′C′* are *A′* (2, −8), *B′* (7, −5), and *C′* (2, −5).

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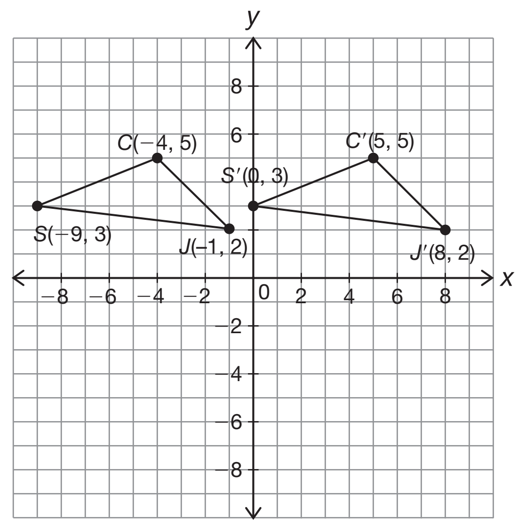
13.3 Using the SSS Congruence Theorem to Identify Congruent Triangles

The Side-Side-Side (SSS) Congruence Theorem states that if three sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent.

Example

Use the SSS Congruence theorem to prove $\triangle CJS$ is congruent to $\triangle C'J'S'$.



$$SC = \sqrt{[-4 - (-9)]^2 + (5 - 3)^2} = \sqrt{5^2 + 2^2} = \sqrt{25 + 4}$$
$$SC = \sqrt{29}$$

$$S'C' = \sqrt{(5 - 0)^2 + (5 - 3)^2} = \sqrt{5^2 + 2^2} = \sqrt{25 + 4}$$
$$S'C' = \sqrt{29}$$

$$CJ = \sqrt{[-1 - (-4)]^2 + (2 - 5)^2} = \sqrt{3^2 + (-3)^2} = \sqrt{9 + 9}$$
$$CJ = \sqrt{18}$$

$$C'J' = \sqrt{(8 - 5)^2 + (2 - 5)^2} = \sqrt{3^2 + (-3)^2} = \sqrt{9 + 9}$$
$$C'J' = \sqrt{18}$$

$$SJ = \sqrt{[-1 - (-9)]^2 + (2 - 3)^2} = \sqrt{8^2 + (-1)^2} = \sqrt{64 + 1}$$
$$SJ = \sqrt{65}$$

$$S'J' = \sqrt{(8 - 0)^2 + (2 - 3)^2} = \sqrt{8^2 + (-1)^2} = \sqrt{64 + 1}$$
$$S'J' = \sqrt{65}$$

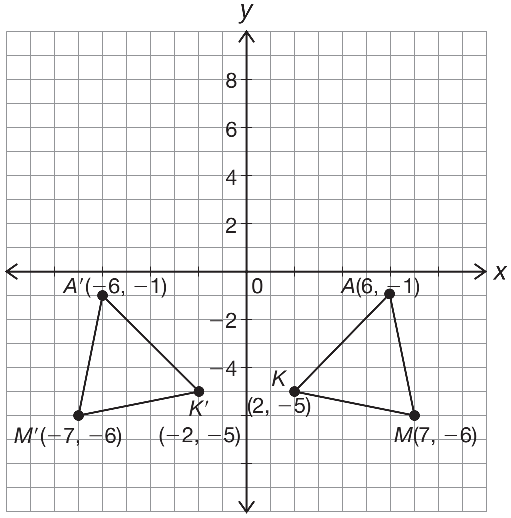
The lengths of the corresponding sides of the pre-image and the image are equal, so the corresponding sides of the image and the pre-image are congruent. Therefore, the triangles are congruent by the SSS Congruence Theorem.

13.4 Using the SAS Congruence Theorem to Identify Congruent Triangles

The Side-Angle-Side (SAS) Congruence Theorem states that if two sides and the included angle of one triangle are congruent to the corresponding two sides and the included angle of a second triangle, then the triangles are congruent. An included angle is the angle formed by two sides of a triangle.

Example

Use the SAS Congruence Theorem to prove that $\triangle AMK$ is congruent to $\triangle A'M'K'$.



$$KA = \sqrt{(6-2)^2 + [-1-(-5)]^2} = \sqrt{4^2 + 4^2} = \sqrt{16+16} = \sqrt{32}$$

$$K'A' = \sqrt{[-5-(-1)]^2 + [-2-(-6)]^2} = \sqrt{(-4)^2 + 4^2} = \sqrt{16+16} = \sqrt{32}$$

$$KM = \sqrt{(7-2)^2 + [-6-(-5)]^2} = \sqrt{5^2 + (-1)^2} = \sqrt{25+1} = \sqrt{26}$$

$$K'M' = \sqrt{[-7-(-2)]^2 + [-6-(-5)]^2} = \sqrt{(-5)^2 + (-1)^2} = \sqrt{25+1} = \sqrt{26}$$

$m\angle K = 58°$

$m\angle K' = 58°$

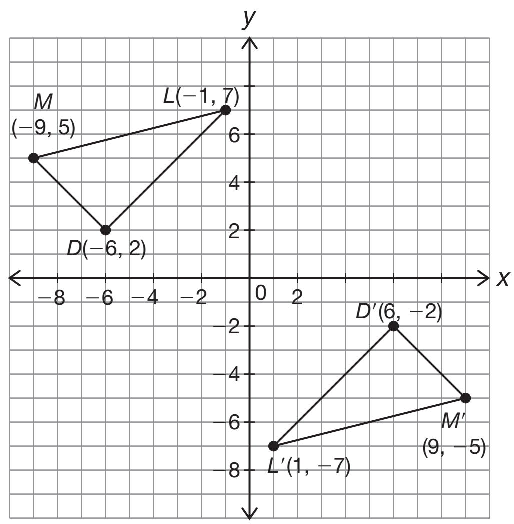
The lengths of the pairs of the corresponding sides and the measures of the pair of corresponding included angles are equal. Therefore, the triangles are congruent by the SAS Congruence Theorem.

13.5 Using the ASA Congruence Theorem to Identify Congruent Triangles

The Angle-Side-Angle (ASA) Congruence Theorem states that if two angles and the included side of one triangle are congruent to the corresponding two angles and the included side of another triangle, then the triangles are congruent. An included side is the line segment between two angles of a triangle.

Example

Use the ASA Congruence Theorem to prove that $\triangle DLM$ is congruent to $\triangle D'L'M'$.



$$DM = \sqrt{[-9-(-6)]^2 + (5-2)^2} = \sqrt{(-3)^2 + 3^2} = \sqrt{9+9} = \sqrt{18}$$

$$D'M' = \sqrt{(9-6)^2 + [-5-(-2)]^2} = \sqrt{3^2 + (-3)^2} = \sqrt{9+9} = \sqrt{18}$$

$m\angle D = 90°$

$m\angle D' = 90°$

$m\angle M = 60°$

$m\angle M' = 60°$

The measures of the pairs of corresponding angles and the lengths of the corresponding included sides are equal. Therefore, the triangles are congruent by the ASA Congruence Theorem.

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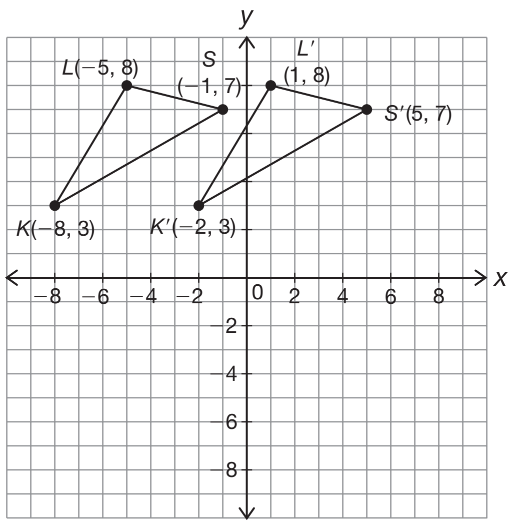
13.6 Using the AAS Congruence Theorem to Identify Congruent Triangles

The Angle-Angle-Side (AAS) Congruence Theorem states that if two angles and a non-included side of one triangle are congruent to the corresponding two angles and the corresponding non-included side of a second triangle, then the triangles are congruent.

Example

Use the AAS Congruence Theorem to prove $\triangle LSK$ is congruent to $\triangle L'S'K'$.



$m\angle L = 108°$

$m\angle L' = 108°$

$m\angle K = 30°$

$m\angle K' = 30°$

$$KS = \sqrt{[-1 - (-8)]^2 + (7 - 3)^2}$$
$$= \sqrt{7^2 + 4^2}$$
$$= \sqrt{49 + 16}$$
$$= \sqrt{65}$$

$$K'S' = \sqrt{[5 - (-2)]^2 + (7 - 3)^2}$$
$$= \sqrt{7^2 + 4^2}$$
$$= \sqrt{49 + 16}$$
$$= \sqrt{65}$$

The measures of the two pairs of corresponding angles and the lengths of the pair of corresponding non-included sides are equal. Therefore, the triangles are congruent by the AAS Congruence Theorem.