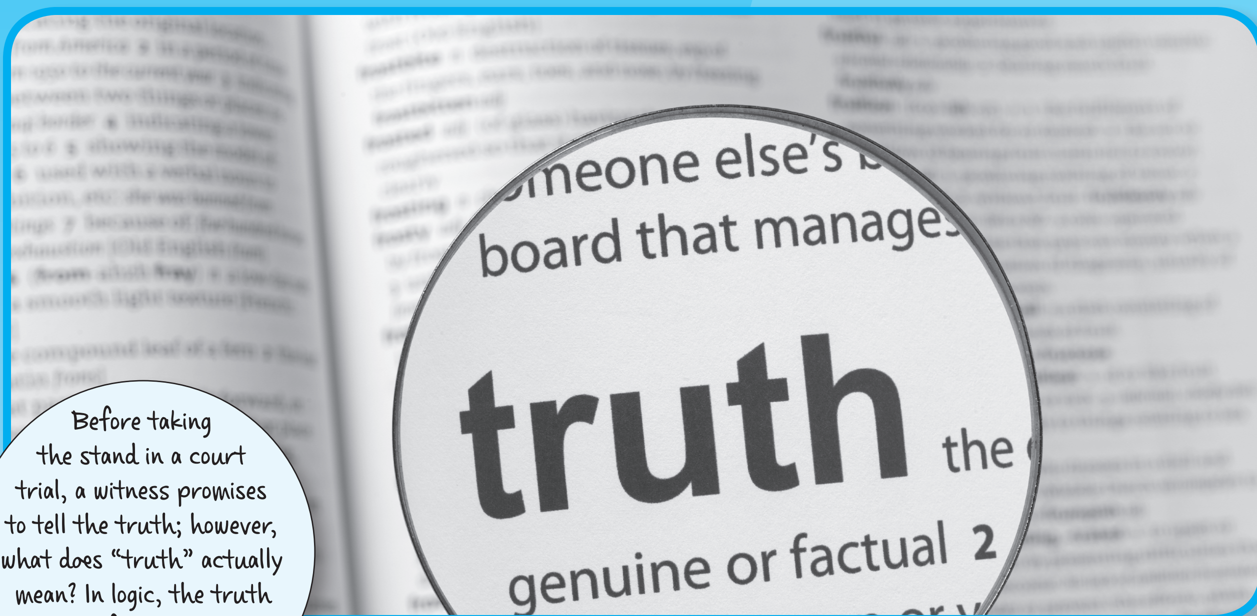
Logic 16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Before taking the stand in a court trial, a witness promises to tell the truth; however, what does "truth" actually mean? In logic, the truth value of a statement might surprise you!

16.1

A Little Dash of Logic

Two Methods of Logical Reasoning

LEARNING GOALS

In this lesson, you will:

- Define inductive reasoning and deductive reasoning.
- Identify methods of reasoning.
- Compare and contrast methods of reasoning.
- Identify why a conclusion may be false.

KEY TERMS

- induction
- deduction

The Greek philosopher Aristotle greatly influenced our understanding of physics, linguistics, politics, and science. He also had a great influence on our understanding of logic. In fact, he is often credited with the earliest study of formal logic, and he wrote six works on logic which were compiled into a collection known as the *Organon*. These works were used for many years after his death. There were a number of philosophers who believed that these works of Aristotle were so complete that there was nothing else to discuss regarding logic. These beliefs lasted until the 19th century when philosophers and mathematicians began thinking of logic in more mathematical terms.

In one of Aristotle's other books, *Metaphysics*, he makes the following statement: "To say that that which is, is not or that which is not is, is a falsehood; and to say that that which is, is and that which is not is not, is true." What is Aristotle trying to say here, and do you agree? Can you prove or disprove this statement?

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PROBLEM 1 How Do You Figure?

1. Emma considered these statements.

   - $4^2 = 4 \times 4$
   - nine cubed is equal to nine times nine times nine
   - 10 to the fourth power is equal to four factors of 10 multiplied together

   Emma concluded that raising a number to a power is the same as multiplying the number as many times as indicated by the exponent. How did Emma reach this conclusion?

2. Ricky read in a book that raising a number to a power is the same as multiplying that number as many times as indicated by the exponent. He had to determine seven to the fourth power using a calculator. So, he entered $7 \times 7 \times 7 \times 7$. How did Ricky reach this conclusion?

3. Compare the reasoning Emma used to the reasoning Ricky used.

4. Jennifer is a writing consultant. She is paid $900 for a ten-hour job and $1980 for a twenty-two-hour job.

   a. How much does Jennifer charge per hour?

   b. To answer Question 4, part (a), did you start with a general rule and make a conclusion, or did you start with specific information and create a general rule?

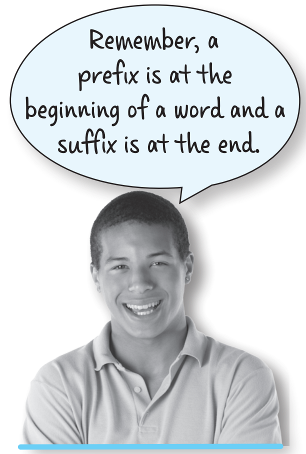
5. Your friend Aaron tutors elementary school students. He tells you that the job pays $8.25 per hour.

   a. How much does Aaron earn if he works 4 hours?

   b. To answer Question 5, part (a), did you start with a general rule and make a conclusion, or did you start with specific information and create a general rule?

PROBLEM 2 Is This English Class or Algebra?

The ability to use information to reason and make conclusions is very important in life and in mathematics. There are two common methods of reasoning. You can construct the name for each method of reasoning using your knowledge of prefixes, root words, and suffixes.



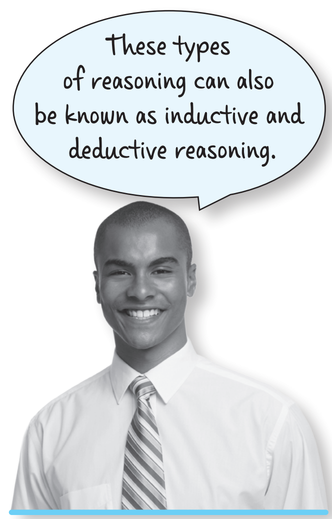
Look at the following information. Remember that a prefix is a beginning of a word, and a suffix is an ending of a word.

- *in*—a prefix that can mean *toward* or *up to*
- *de*—a prefix that can mean *down from*
- *duc*—a base or root word meaning *to lead* and often *to think*, from the Latin word *duco*
- *-tion*—a suffix that forms a noun meaning *the act of*

1. Form a word that means "the act of thinking down from."

2. Form a word that means "the act of thinking toward or up to."

Induction is reasoning that involves using specific examples to make a conclusion. Many times in life you must make generalizations about observations or patterns and apply these generalizations to new or unfamiliar situations. For example, you may notice that when you don't study for a test, your grade is lower than when you do study for a test. You apply what you learned from these observations to the next test you take.



Deduction is reasoning that involves using a general rule to make a conclusion. For example, you may learn the rule for which direction to turn a screwdriver: "righty tighty, lefty loosey." If you want to unscrew a screw, you apply the rule and turn the screwdriver counterclockwise.

3. Refer back to Problem 1.

a. Identify which Question(s) that used inductive reasoning.

b. Identify which Question(s) that used deductive reasoning.

PROBLEM 3 Coming to Conclusions

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A problem situation can provide you with a great deal of information which you can use to make conclusions. It is important to identify specific and general information in a problem situation in order to come to appropriate conclusions.

Ms. Ross teaches an Economics class every day from 1:00 to 2:15. She has noticed that Andrew has not turned in his homework 3 days this week. She is concerned that Andrew's grade will fall if he does not turn in his homework.

General information:
Ms. Ross teaches an Economics class every day from 1:00 to 2:15.

Specific information:
Andrew has not turned in his homework 3 days this week.

Conclusion:
Andrew's grade will fall if he does not turn in his homework.

1. Did Ms. Ross use induction or deduction to make this conclusion? Explain your answer.

2. Conner read a newspaper article that states that the use of tobacco greatly increases the risk of getting cancer. He then notices that his neighbor Matilda smokes. Connor is concerned that Matilda has a high risk of getting cancer.

   a. Which information is specific and which information would be considered general in this problem situation?

   b. What is the conclusion in this problem?

   c. Did Conner use inductive or deductive reasoning to make the conclusion? Explain your reasoning.

d. Is Conner's conclusion correct? Explain your reasoning.

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3. Molly returned from a trip to England and tells you, "It rains every day in England!" She explains that it rained each of the five days she was there.

a. Which information is specific and which information would be considered general in this problem situation?

b. What is the conclusion in this problem?

c. Did Molly use inductive or deductive reasoning to make the conclusion? Explain your answer.

d. Is Molly's conclusion correct? Explain how you determine whether Molly is correct.

4. Dontrell takes detailed notes in history class and math class. His classmate Trang will miss biology class tomorrow to attend a field trip. Trang's biology teacher asks Trang if he knows someone in class who always takes detailed notes. Trang tells his biology teacher that Dontrell takes detailed notes. Trang's biology teacher suggests Trang borrow Dontrell's notes because the teacher concludes that Dontrell's notes will be detailed.

a. What conclusion did Trang make? What information supports this conclusion?

b. What type of reasoning did Trang use? Explain your reasoning.

c. What conclusion did the biology teacher make? What information supports this conclusion?

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d. What type of reasoning did the biology teacher use? Explain your reasoning.

e. Will Trang's conclusion always be true? Will the biology teacher's conclusion always be true? Explain your reasoning.

5. The first four numbers in a sequence are 4, 15, 26, and 37.

a. What is the next number in the sequence? How did you calculate the next number?

b. Describe how you used both induction and deduction, and what order you used these reasonings to make your conclusion.

6. Write a short note to a friend explaining induction and deduction. Include definitions of both terms and examples that are very easy to understand.

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PROBLEM 4 Why Is This False?

There are two reasons why a conclusion may be false. Either the assumed information is false, or the argument is not valid.

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1. Derek tells his little brother that it will not rain for the next 30 days because he "knows everything." Why is this conclusion false?

2. Two lines are not parallel, so the lines must intersect. Why is this conclusion false?

3. Write an example of a conclusion that is false because the assumed information is false.

4. Write an example of a conclusion that is false because the argument is not valid.

Be prepared to share your solutions and methods.

What's Your Conclusion?

16.2

Understanding Conditional Statements, Arguments, and Truth Tables

LEARNING GOALS

In this lesson, you will:

- Define a conditional statement.
- Identify the hypothesis and conclusion of a conditional statement.
- Explore the truth value of conditional statements.
- Use a truth table.

KEY TERMS

- conditional statement
- propositional form
- propositional variables
- hypothesis
- conclusion
- truth value
- truth table
- converse
- inverse
- contrapositive
- logically equivalent
- biconditional statement

You have probably learned about "if-then" statements in language arts. You are now going to use "if-then" statements while studying logic. But did you also know that "if-then" statements are used in computer programming? These statements tell the computer program to perform different actions depending on whether the "if-then" statement is true or false. There can also be an "else" branch which gives the computer an alternative action to complete.

An example of this "if-then-else" programming code may be shown as follows.

```
If count = 0 Then
message = "There are no items."
Elseif count = 1 Then
message = "There is 1 item."
Elseif count = >1 Then
message = "There are" & count & "items."
```

Can you decipher this piece of computer programming language? What would be happening on the computer if it was reading this piece of code?

PROBLEM 1 What's Your Conclusion?

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Read each pair of statements. Then write a valid conclusion.

1. Statement: Melanie's school guidance counselor tells her that if she applies for a scholarship, then she will have a chance to receive it.

   Statement: Melanie applies for a scholarship.

   Conclusion:

2. Statement: If it rains, then the baseball game will be cancelled.

   Statement: It rains.

   Conclusion:

3. Statement: If Suzanne misses the application deadline for the Vocational Training School, then she will not be admitted.

   Statement: Suzanne missed the application deadline.

   Conclusion:

4. Statement: Marvin will know whether he enjoys waltz lessons if he attends his first waltz lesson.

   Statement: Marvin attended his first waltz lesson.

   Conclusion:

5. Statement: If having the most experience as a nuclear engineer had been the main requirement, then Olga would have gotten the job.

   Statement: Olga did not get the job.

   Conclusion:

Read each statement and conclusion. Then write the additional statement required to reach the conclusion.

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6. Statement: If no evidence is found linking a suspect to the scene of a crime, then the suspect will be found innocent.

Statement: ______________________________

Conclusion: Therefore, the suspect was found innocent.

7. Statement: If the community service program chooses the litter removal project, then Mayor Elder will have the carnival in the North Shore neighborhood.

Statement: ______________________________

Conclusion: Therefore, the community service program did not choose the litter removal project.

8. Statement: If tulips are to survive, then they need sunlight.

Statement: ______________________________

Conclusion: Therefore, the tulips did not survive.

9. Statement: The Secret Service will be at the dinner if the President shows up.

Statement: ______________________________

Conclusion: Therefore, the President did not show up at the dinner.

10. Statement: ______________________________

Statement: You have your umbrella.

Conclusion: Therefore, it must be raining.

11. Statement: ______________________________

Statement: You ate a good breakfast.

Conclusion: Therefore, you were not hungry before lunch.

12. Statement: ______________________________

Statement: You did your math homework.

Conclusion: Therefore, your teacher was happy.

A **conditional statement** is a statement that can be written in the form "If p, then q." The form of this conditional statement is called the **propositional form**. The propositional form can also be written using symbols as $p \rightarrow q$ and is read as "p implies q." The variables p and q are **propositional variables**. The **hypothesis** of a conditional statement is the variable p, and the **conclusion** of a conditional statement is the variable q.

The **truth value** of a conditional statement is whether the statement is true or false. If a conditional statement *could* be true, then the truth value of the statement is considered true. The truth value of a conditional statement is either true or false, but not both.

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PROBLEM 2 If This, Then That

Consider the conditional statement:

If the measure of an angle is 32°, then the angle is acute.

Use the conditional statement to answer each question.

1. What is the hypothesis p?

2. What is the conclusion q?

3. When p is true and q is true, the truth value of a conditional statement is true.
   a. What does the phrase "p is true" mean in terms of the given conditional statement?

   b. What does the phrase "q is true" mean in terms of the given conditional statement?

   c. Explain why the truth value of the given conditional statement is true.

4. When *p* is true and *q* is false, the truth value of a conditional statement is false.

a. What does the phrase "*p* is true" mean in terms of the given conditional statement?

b. What does the phrase "*q* is false" mean in terms of the given conditional statement?

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c. Explain why the truth value of the given conditional statement is false.

5. When *p* is false and *q* is true, the truth value of a conditional statement is true.

a. What does the phrase "*p* is false" mean in terms of the given conditional statement?

b. What does the phrase "*q* is true" mean in terms of the given conditional statement?

c. Explain why the truth value of the given conditional statement is true.

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6. When p is false and q is false, the truth value of a conditional statement is true.

a. What does the phrase "p is false" mean in terms of the given conditional statement?

b. What does the phrase "q is false" mean in terms of the given conditional statement?

c. Explain why the truth value of the given conditional statement is true.

A **truth table** is a table that summarizes all possible truth values for a conditional statement $p \rightarrow q$. The first two columns of a truth table represent all possible truth values for the propositional variables p and q. The last column represents the truth value of the conditional statement $p \rightarrow q$.

The truth value for the conditional statement "If the measure of an angle is 32°, then the angle is acute" is shown.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

7. Consider the conditional statement "If $m\overline{AB} = 6$ inches and $m\overline{BC} = 6$ inches, then $\overline{AB} \cong \overline{BC}$."

a. What is the hypothesis *p*?

b. What is the conclusion *q*?

c. When both *p* and *q* are true, what does that mean? What is the truth value of the conditional statement when both *p* and *q* are true?

d. When *p* is true and *q* is false, what does that mean? What is the truth value of the conditional statement when *p* is true and *q* is false?

e. When *p* is false and *q* is true, what does that mean? What is the truth value of the conditional statement when *p* is false and *q* is true?

f. When both *p* and *q* are false, what does that mean? What is the truth value of the conditional statement when both *p* and *q* are false?

PROBLEM 3 Converses

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The **converse** of a conditional statement of the form "If *p*, then *q*" is the statement of the form "If *q*, then *p*." The converse is a new statement that results when the hypothesis and conclusion of the conditional statement are switched.

For each conditional statement written in propositional form, identify the hypothesis *p* and conclusion *q*. Then switch them to write the converse of the conditional statement.

1. If a quadrilateral is a square, then the quadrilateral is a rectangle.
   a. Hypothesis, *p*:

   b. Conclusion, *q*:

   c. What is the truth value of this conditional statement? Explain your reasoning.

   d. Converse:

   e. What is the truth value of this converse? Explain your reasoning.

2. If an integer is even, then the integer is divisible by two.
   a. Hypothesis, *p*:

   b. Conclusion, *q*:

   c. What is the truth value of this conditional statement? Explain your reasoning.

   d. Converse:

   e. What is the truth value of this converse? Explain your reasoning.

3. If a polygon has six sides, then the polygon is a pentagon.

   a. Hypothesis, *p*:

   b. Conclusion, *q*:

   c. What is the truth value of this conditional statement? Explain your reasoning.

   d. Converse:

   e. What is the truth value of this converse? Explain your reasoning.

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4. If two lines intersect, then the lines are perpendicular.

   a. Hypothesis, *p*:

   b. Conclusion, *q*:

   c. What is the truth value of this conditional statement? Explain your reasoning.

   d. Converse:

   e. What is the truth value of this converse? Explain your reasoning.

5. What do you notice about the truth value of a conditional statement and the truth value of its converse?

PROBLEM 4 Inverses

The **inverse** of a conditional statement of the propositional form "If p, then q" is the statement of the form "If not p, then not q." The inverse is a new statement that results when the hypothesis and conclusion of the conditional statement are negated.

Analyze each conditional statement and determine its truth value. Then identify the negation of the hypothesis and the conclusion. Finally write the inverse of the conditional statement.

1. If a quadrilateral is a square, then the quadrilateral is a rectangle.

   a. What is the truth value of this conditional statement? Explain your reasoning.

   b. Inverse:

   c. What is the truth value of the inverse? Explain your reasoning.

2. If an integer is even, then the integer is divisible by two.

   a. What is the truth value of this conditional statement? Explain your reasoning.

   b. Not p:

   c. Not q:

   d. Inverse:

   e. What is the truth value of the inverse? Explain your reasoning.

3. If a polygon has six sides, then the polygon is a pentagon.

   a. What is the truth value of this conditional statement? Explain your reasoning.

   b. Not *p*:

   c. Not *q*:

   d. Inverse:

   e. What is the truth value of the inverse? Explain your reasoning.

4. If two lines intersect, then the lines are perpendicular.

   a. What is the truth value of this conditional statement? Explain your reasoning.

   b. Not *p*:

   c. Not *q*:

   d. Inverse:

   e. What is the truth value of the inverse? Explain your reasoning.

5. What do you notice about the truth value of a conditional statement and the truth value of its inverse?

PROBLEM 5 Contrapositives

The **contrapositive** of a conditional statement of the propositional form "If p, then q" is the statement of the form "If not q, then not p." The contrapositive is a new statement that results when the hypothesis and conclusion of the conditional statement are negated and switched.

Analyze each conditional statement and determine its truth value. Then write the contrapositive of the conditional statement and the truth value of the contrapositive statement.

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1. If a quadrilateral is a square, then the quadrilateral is a rectangle.
   a. What is the truth value of this conditional statement? Explain your reasoning.

   b. Contrapositive:

   c. What is the truth value of this contrapositive? Explain your reasoning.

2. If an integer is even, then the integer is divisible by two.
   a. What is the truth value of this conditional statement? Explain your reasoning.

   b. Contrapositive:

   c. What is the truth value of this contrapositive? Explain your reasoning.

3. If a polygon has six sides, then the polygon is a pentagon.

   a. What is the truth value of this conditional statement? Explain your reasoning.

   b. Contrapositive:

   c. What is the truth value of this contrapositive? Explain your reasoning.

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4. If two lines intersect, then the lines are perpendicular.

   a. What is the truth value of this conditional statement? Explain your reasoning.

   b. Contrapositive:

   c. What is the truth value of this contrapositive? Explain your reasoning.

5. What do you notice about the truth value of a conditional statement and the truth value of its contrapositive?

PROBLEM 6 So Is It True?

1. For each statement, tell whether you agree or disagree. If you disagree, provide a counterexample.

   a. If a conditional statement is true, then its converse is true.

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   b. If a conditional statement is true, then its inverse is true.

   c. If a conditional statement is true, then its contrapositive is true.

Two propositional forms are **logically equivalent** if they have the same truth values for corresponding values of the propositional variables.

2. Look at the four conditional statements used in Problems 2 through 4. Which conditional statement contained the most examples of logically equivalent relationships?

The negation of a statement p is logically equivalent to the statement "It is not true that p." The negation of a statement p is represented as "not p" or $\sim p$.

3. If the truth value of p is "true," what is the truth value of $\sim p$?

4. If the truth value of p is "false," what is the truth value of $\sim p$?

5. Complete the following truth table.

				Conditional	
p	$\sim p$	q	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$
T		T			
T		F			
F		T			
F		F			

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6. What do you notice about the last two columns?

7. The truth table proves that a conditional statement is logically equivalent to what other propositional form?

PROBLEM 7 Biconditional Statements

When a conditional statement and its converse are both true, they can be combined and written as a single statement using "if and only if." This new statement is called a **biconditional statement**.

Conditional Statement: If a quadrilateral has four right angles, then the quadrilateral is a rectangle.

Converse: If a quadrilateral is a rectangle, then the quadrilateral has four right angles.

The conditional statement and its converse are both true. So, they can be rewritten as a biconditional statement.

Biconditional: A quadrilateral has four right angles if and only if the quadrilateral is a rectangle.

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For each conditional statement, write the converse. If possible, write a true biconditional statement. If it is not possible, explain why.

1. If a triangle has at least two congruent sides, then the triangle is isosceles.

   a. Converse:

   b. Biconditional:

2. If two lines are parallel, then the two lines do not intersect.

   a. Converse:

   b. Biconditional:

3. If two circles have equal length radii, then the two circles are congruent.

   a. Converse:

   b. Biconditional:

4. If a quadrilateral is a square, then the quadrilateral is a rectangle.

   a. Converse:

   b. Biconditional:

5. If an angle is bisected, then the angle is divided into two angles of equal measure.

   a. Converse:

   b. Biconditional:

Be prepared to share your solutions and methods.

Proofs Aren't Just for Geometry

16.3

Introduction to Direct and Indirect Proof with the Properties of Numbers

LEARNING GOALS

In this lesson, you will:

- Use the commutative, associative, identity, and inverse properties for addition and multiplication.
- Use the distributive property.
- Use direct proof to prove a theorem.
- Use indirect proof to prove a theorem.

KEY TERM

- proof by contradiction

Throughout the day your teachers, friends, and other students are probably supplying you with a lot of new information. When they give you this new information, do you believe it or do you ask them to prove it? Hopefully, unless it is something you know to be a fact, you ask them to prove it. The word proof comes from the Latin word *probare* which means "to test." It used to be that people of authority were to be believed without having to provide proof. Luckily today this is not the case. You are able to ask for proof when given information, and this is especially important when dealing with math and science as there are many new ideas and theories being presented all the time.

Do you think all the information you learn throughout the day is true? How could you prove or disprove the information? Why is it important to ask for proof instead of just believing everything you hear without question?

PROBLEM 1 Direct Proofs and Number Laws

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1. Some conditional statements can be proven using a direct proof. Read the conditional statement and each step of the direct proof. For each step, explain what changed from the previous step.

Conditional Statement: If $a + bc = c(b + a) + a$, then $a = 0$ or $c = 0$.

Steps	What Changed?
$a + bc = cb + ca + a$	____________________
$a + bc = bc + ca + a$	____________________
$a + bc = ca + a + bc$	____________________
$a + bc - bc = ca + a + bc - bc$	____________________
$a = ca + a$	____________________
$a - a = ca + a - a$	____________________
$0 = ca$	____________________
$a = 0$ or $c = 0$	____________________

2. Complete the table to summarize the real number properties, $\forall\ a, b \in \mathbb{R}$.

Name of Property	Symbolic Representation of Property Under Addition	Symbolic Representation of Property Under Multiplication
Commutative	$a + b = b + a$	
Associative		
Identity		
Inverse		$a \cdot \frac{1}{a} = 1 \ \ (a \neq 0)$
Distributive		

3. Refer to the direct proof from Question 1. Use the names of the properties from the table. If you cannot find a property that is a good fit, write a statement that summarizes the rule or property of real numbers.

Conditional Statement: If $a + bc = c(b + a) + a$, then $a = 0$ or $c = 0$.

Steps	**Reasons**
$a + bc = cb + ca + a$	
$a + bc = bc + ca + a$	
$a + bc = ca + a + bc$	
$a + bc - bc = ca + a + bc - bc$	
$a = ca + a$	
$a - a = ca + a - a$	
$0 = ca$	
$a = 0$ or $c = 0$	

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Consider the associative properties $(a + b) + c = a + (b + c)$, and $a(bc) = (ab)c$.

The associative property can be stated in words shown.

When three terms are added, the first two terms can be grouped or the last two terms can be grouped.

When three factors are multiplied, the first two factors can be grouped or the last two factors can be grouped.

4. State the commutative property in words.

5. State the identity property in words.

6. State the inverse property in words.

7. State the distributive property in words.

PROBLEM 2 Indirect Proofs

In Problem 1, you used a direct proof to prove the theorem "If $a + bc = c(b + a) + a$, then $a = 0$ or $c = 0$." This theorem can also be proven using an indirect proof called **proof by contradiction**.

To prove a statement using proof by contradiction, assume that the conclusion is false. Then show that the hypothesis is false or state a contradiction. This is equivalent to showing that if the hypothesis is true, then the conclusion is also true.

In the theorem in Problem 1, the conclusion was $a = 0$ or $c = 0$. Now, begin by assuming that $a \neq 0$ and $c \neq 0$. So, let $a = 2$ and $c = 2$. Substitute these values into the equation and simplify.

1. Complete the indirect proof. Use the names of the real number properties.

Steps	Reasons
$2 + 2b = 2(b + 2) + 2$	Assumption (negation of the conclusion)
$2 + 2b = 2b + 4 + 2$	
$2 + 2b = 4 + 2 + 2b$	
$0 = 4$	

2. Is this a contradiction? What does this tell you about the theorem?

3. Katie determines that the statement $\frac{ab + c}{a} = b + c$ is true for all real numbers a, b, and c. Is she correct? Show or explain your reasoning using proof by contradiction.

4. Prove or disprove the statement $\frac{ab + ac}{a} = b + c$.

5. Lamar proved that if $a = 0$, then $5 = 7$ using the steps shown. While he knows he is incorrect, he cannot identify his error.

Lamar

If $a = 0$, then $5 = 7$.

1.	$0 = 0$	All numbers equal themselves.
2.	$5 \cdot 0 = 7 \cdot 0$	Zero times any number is equal to zero.
3.	$5a = 7a$	Let $a = 0$.
4.	$5a + ax = 7a + ax$	Algebraic equations remain true if you perform the same operation on both sides.
5.	$a(5 + x) = a(7 + x)$	Distributive law of multiplication with respect to addition.
6.	$\frac{a(5 + x)}{a} = \frac{a(7 + x)}{a}$	Algebraic equations remain true if you perform the same operation on both sides.
7.	$5 + x = 7 + x$	Inverse law of multiplication
8.	$5 = 7$	Algebraic equations remain true if you perform the same operation on both sides.

Identify the error in Lamar's proof.

Be prepared to share your solutions and methods.

16.4

Your Oldest Likes Spinach?

Using Logic to Solve Problems, Part 1

LEARNING GOALS

In this lesson, you will:

- Solve problems using logic.
- Solve logic problems using grids.

"How often have I said to you that when you have eliminated the impossible, whatever remains, *however improbable*, must be the truth? We know that he did not come through the door, the window, or the chimney. We also know that he could not have been concealed in the room, as there is no concealment possible. When, then, did he come?"

This quote comes from a novel called *The Sign of the Four* by Sir Arthur Conan Doyle, and it features the famous detective Sherlock Holmes. Holmes and his companion Watson used logic to solve mysterious cases in which crimes were committed. Logical thinking is often used by detectives and police officers to solve crimes, but most of us also use logic every day. How do you use logical thinking in mathematics? How else might you use logical thinking?

PROBLEM 1 Can I Ask You a Few Questions?

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A census taker was going from door to door in a neighborhood, gathering information on the number of people living at each address and their ages. He came to the door of a math teacher who answered the knock. The census taker and the math teacher had the following conversation.

Census Taker: Good morning, Miss! I am gathering information for the census. How many people are living at this address?

Math Teacher: I live here with my husband and three children.

Census Taker: Great! What are your children's ages?

Math Teacher: Well, since I teach math and I love creating mathematical riddles, I am going to ask you to figure out their ages. First, let me tell you that the product of their ages is 72.

Census Taker: That's fine, but I still need more information.

Math Teacher: Okay, the sum of their ages is the same as the street address number of my house.

Census Taker: Oh, I see!

At this point, the census taker looked at the number on the house and thought for a couple of minutes.

Census Taker: I see the house number, but unfortunately I still need more information.

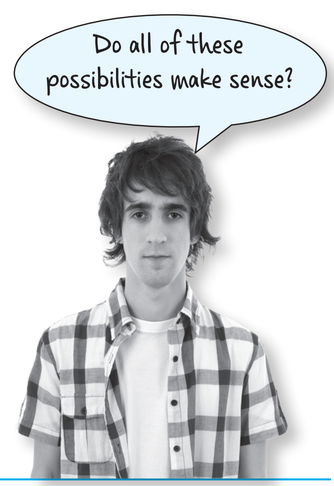
Math Teacher: That's true. You do still need more information. Okay, my oldest child really likes spinach.

After thinking about this new information for a minute, the census taker recorded the children's ages. Then the census taker asked for the math teacher's age and her husband's age and recorded these as well. The census taker thanked the math teacher and went on his way.

1. How did the census taker figure out the children's ages from the information the math teacher provided?

2. What is the first clue the math teacher gave?

3. List all the possible ages for the children.



4. What is the second clue?

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5. Determine the sums of all the possible ages for the children.

6. After hearing the second clue, the census taker said he needed more information, and the math teacher agreed. Why was it true that the census taker still needed more information? What does this tell you about the possibilities?

7. What is the final clue?

8. How does this clue enable you to determine the children's ages?

9. What are the children's ages? Does your answer fit with the clues?

PROBLEM 2 Computer Games

Jose, Katie, and Sidney were playing computer games on different computers. One student used the laptop computer to play Sweepminer. Jose played a card game, while another student played pinball. Neither Jose nor Sidney used the netbook. Sidney did not use the desktop but played Sweepminer. If each student played one game on one type of computer, which student used which computer and what game did each student play?

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This is an example of a type of problem called a logic puzzle. To solve a logic puzzle, a grid is often used to keep the information organized.

1. List each clue.
   - One student playing Sweepminer used the laptop computer.
   - ____________________
   - ____________________
   - ____________________

2. List the different categories and members of each category.
   - Students: Jose, Katie, and Sidney
   - ____________________
   - ____________________

Since there are three categories with three possible members, you will use a 3 × 3 logic puzzle grid.

		Category 2			Category 3		
		Member 1	Member 2	Member 3	Member 1	Member 2	Member 3
Category 1	Member 1						
	Member 2						
	Member 3						
Category 3	Member 1						
	Member 2						
	Member 3						

Next, identify each category and list the members in each category.

		Computers			Games		
		Laptop	Netbook	Desktop	Sweepminer	Card Game	Pinball
Students	Jose						
	Katie						
	Sidney						
Games	Sweepminer						
	Card Game						
	Pinball						

Once the grid is set up, use it to keep track of the clues by placing an X in a cell that is not true and an O in a cell that is true.

3. The first clue: One student playing Sweepminer used the laptop computer.

Locate the cell where the laptop column intersects with the Sweepminer row and place an O in the cell.

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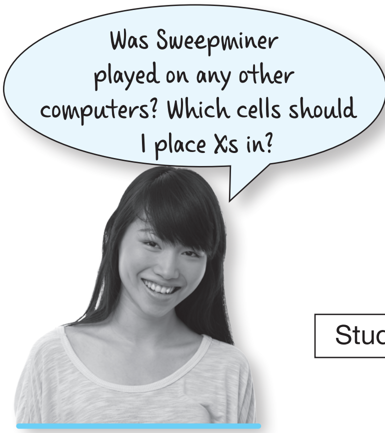


		Computers			Games		
		Laptop	Netbook	Desktop	Sweepminer	Card Game	Pinball
Students	Jose						
	Katie						
	Sidney						
Games	Sweepminer						
	Card Game						
	Pinball						

4. The second clue: Jose played a card game, while another student played pinball.

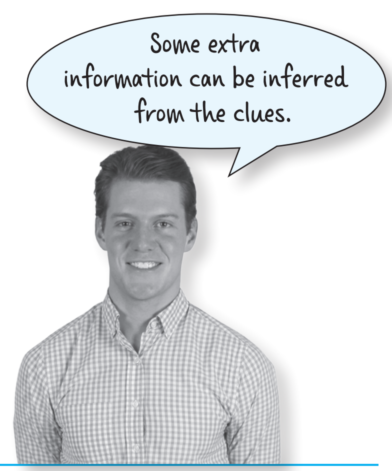
Use this clue to place an X or O in the appropriate cells in the grid.

5. The third clue: Neither Jose nor Sidney used the netbook.

Use this clue to place an X or O in the appropriate cells in the grid.

6. The fourth clue: Sidney did not use the desktop, but played Sweepminer.

Use this clue to place an X or O in the appropriate cells in the grid.



7. Finish placing X's and O's in the cells to complete the grid. Then, complete each sentence.

a. Sidney used the ______________ (computer) and played ______________ (game).

b. Katie used the ______________ (computer) and played ______________ (game).

c. Jose used the ______________ (computer) and played ______________ (game).

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8. Explain how you were able to complete the grid.

PROBLEM 3 Favorite Sports

Matthew, Deanna, and Ashley are friends and they each go to three different middle schools: Washington, Jefferson, and Carver. Each friend likes a different sport: football, soccer, or basketball. Matthew likes football. Deanna does not attend Jefferson. The girl who attends Jefferson likes basketball. The friend who attends Carver likes soccer. Who goes to which school and likes what sport?

1. List the categories and the members of each.

Category 1: Members:

Category 2: Members:

Category 3: Members:

2. List the clues.

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3. Complete the grid.

		Washington	Jefferson	Carver	Football	Soccer	Basketball
	Matthew						
	Deanna						
	Ashley						
	Football						
	Soccer						
	Basketball						

4. Who goes to which school and likes what sport?

Be prepared to share your solutions and methods.

16.5

Shoes and Math Scores?

Using Logic to Solve Problems, Part 2

LEARNING GOALS

In this lesson, you will:

- Solve problems using logic.
- Solve logic problems using grids.

You see four cards. Each card has a letter on one side and a number on the other side. The letters and numbers you see right now are D, K, 3, and 7.

D	K	3	7

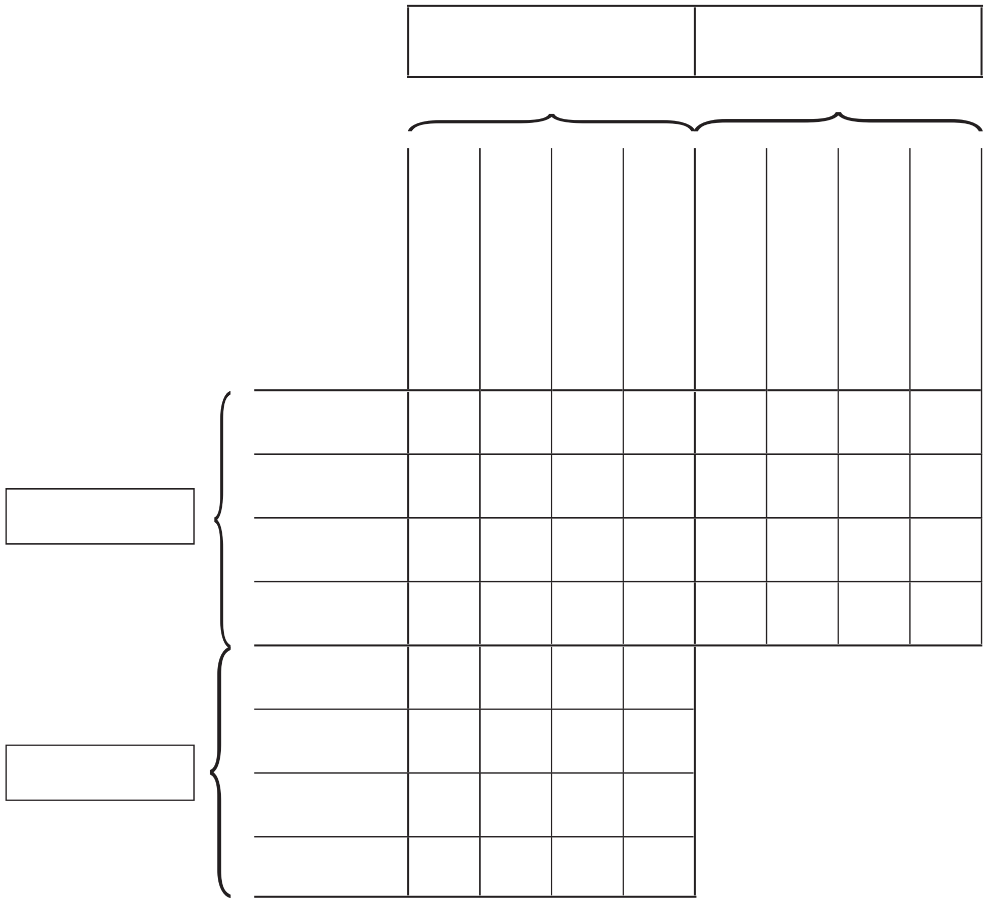
Here is a rule: Every card that has a D on one side has a 3 on the other side. Which cards must you turn over to determine if the rule is true? This logical reasoning task is known as the Wason Selection Task. Research published in 2004 found that only about 30% of college students studying math came up with the correct answer. What is the correct answer?

PROBLEM 1 Shoes, Boots, and Sandals

Kane, Toni, Darlene, and Brenda shop at The Athlete's Shoe. Each buys a different type of shoes. The Athlete's Shoe sells running shoes, cross-trainers, hiking boots, and sandals. The prices of the shoes are $35, $40, $45, and $50. Who bought which type of shoe, and how much did each person pay. Use the given clues.

- Kane did not buy sandals.
- Toni bought sandals.
- Darlene did not buy running shoes, sandals, or hiking boots.
- Brenda hates hiking.
- The sandals cost $40.
- The hiking boots are the most expensive.
- The running shoes do not cost $45.

1. Complete the grid.



2. Who bought which shoes, and how much did each person pay?

PROBLEM 2 The Highest Scores

On the last math test, there were five students who had the highest scores: three boys, Herbert, Albert, and Antonio, and two girls, Angela and Leticia. Each student wore a different color of shirt: green, brown, blue, pink, or red. Which student wore which color, and what was the score ranking he or she received? Use the clues given.

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- The girls wore colors that had five letters in their name.
- The student in blue had the highest of the five scores, and the student in green had the lowest of the five scores.
- Neither Leticia nor Antonio got the fourth-highest score.
- Neither Leticia nor Antonio got the highest or lowest scores, and Herbert did not wear pink.
- The student in pink got the score right below the person in blue.
- Albert wore red.

1. Complete the grid.

2. Which student wore which color, and what was the score ranking he or she received?

PROBLEM 3 Favorite Hobbies and Foods

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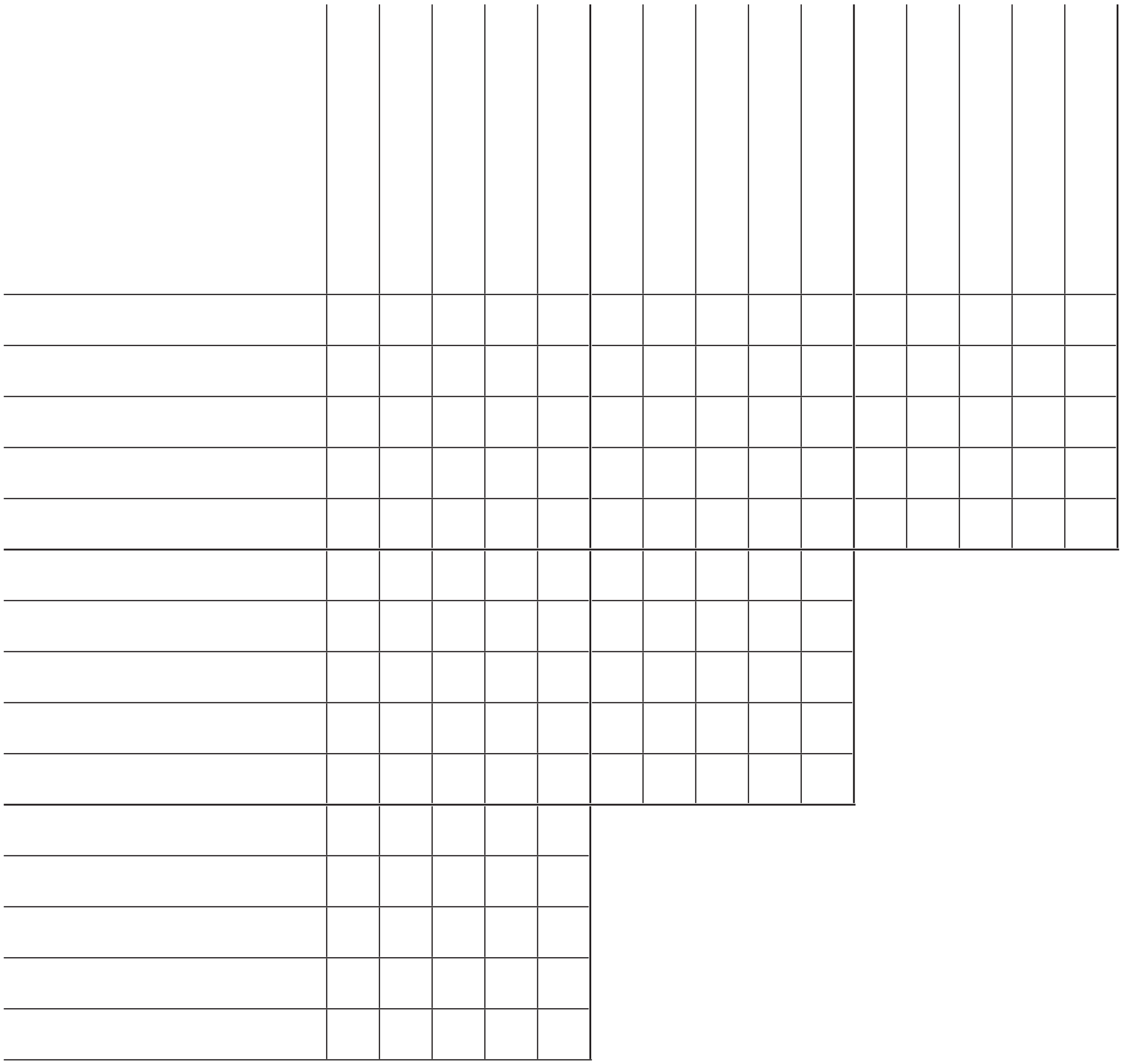
Abbey, Lois, Mindy, Toni, and Xerxes are all friends who like to do their own thing. Their ages are 60, 50, 40, 23, and 20. Each friend likes one of the following hobbies: watching television, playing computer games, reading, playing board games, and writing poetry. Each friend also has a favorite food from the following list: fish, salad, chicken, spinach, and apples. What is each person's age, hobby, and favorite food? Use the clues given.

- Toni does not know how to operate a computer.
- Abbey loves salad.
- The oldest person loves to watch television.
- The name of the 20-year-old does not start with the letter M or X.
- Lois likes chicken.
- The youngest person likes to read.
- Xerxes is 10 years older than Toni.
- The person who is 40 years old likes to play computer games.
- The person who likes to watch TV likes to eat fish.
- Toni likes spinach.
- Toni does not write poetry.
- Abbey is younger than Lois.
- Lois does not like to play board games.
- Lois is not older than 29.
- Mindy loves apples.
- Mindy is twenty years older than Abbey.

1. Complete the grid.



2. What is each person's age, hobby, and favorite food?

Be prepared to share your solutions and methods.

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Chapter 16 Summary

KEY TERMS

- induction (16.1)
- deduction (16.1)
- conditional statement (16.2)
- propositional form (16.2)
- propositional variables (16.2)
- hypothesis (16.2)
- conclusion (16.2)
- truth value (16.2)
- truth table (16.2)
- converse (16.2)
- inverse (16.2)
- contrapositive (16.2)
- logically equivalent (16.2)
- biconditional statement (16.2)
- proof by contradiction (16.3)

16.1 Identifying and Comparing Induction and Deduction

Induction uses specific examples to make a conclusion. Induction, also known as inductive reasoning is used when observing data, recognizing patterns, making generalizations about the observations or patterns, and reapplying those generalizations to unfamiliar situations. Deduction, also known as deductive reasoning, uses a general rule or premise to make a conclusion. It is the process of showing that certain statements follow logically from some proven facts or accepted rules.

Example

Kyra sees coins at the bottom of a fountain. She concludes that if she throws a coin into the fountain, it too will sink. Tyler understands the physical laws of gravity and mass and decides a coin he throws into the fountain will sink.

The specific information is the coins Kyra and Tyler observed at the bottom of the fountain.
The general information is the physical laws of gravity and mass.
Kyra's conclusion that her coin will sink when thrown into the fountain is induction.
Tyler's conclusion that his coin will sink when thrown into the fountain is deduction.

16.1 Identifying False Conclusions

It is important that all conclusions be tracked back to given truths. There are two reasons why a conclusion may be false. Either the assumed information is false or the argument is not valid.

Example

Erin noticed that every time she missed the bus, it rained. So, she concludes that next time she misses the bus it will rain.

Erin's conclusion is false because missing the bus is not related to what makes it rain.

16.2 Writing a Conditional Statement

A conditional statement is a statement that can be written in the form "If *p*, then *q*." The portion of the statement represented by *p* is the hypothesis. The portion of the statement represented by *q* is the conclusion.

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Example

If I plant an acorn, then an oak tree will grow.

A solid line is drawn under the hypothesis, and a dotted line is drawn under the conclusion.

Statement: If I plant an acorn, then an oak tree will grow.

Statement: No oak tree grew.

Conclusion: Therefore, I did not plant an acorn.

16.2 Using a Truth Table to Explore the Truth Value of a Conditional Statement

The truth value of a conditional statement is whether the statement is true or false. If a conditional statement could be true, then its truth value is considered "true." The first two columns of a truth table represent the possible truth values for *p* (the hypothesis) and *q* (the conclusion). The last column represents the truth value of the conditional statement ($p \rightarrow q$). Notice that the truth value of a conditional statement is either "true" or "false," but not both.

Example

Consider the conditional statement, "If I eat too much, then I will get a stomach ache."

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

When *p* is true, I ate too much. When *q* is true, I will get a stomach ache. It is true that when I eat too much, I will get a stomach ache. So, the truth value of the conditional statement is true.

When *p* is true, I ate too much. When *q* is false, I will not get a stomach ache. It is false that when I eat too much, I will not get a stomach ache. So, the truth value of the conditional statement is false.

When *p* is false, I did not eat too much. When *q* is true, I will get a stomach ache. It could be true that when I did not eat too much, I will get a stomach ache for a different reason. So, the truth value of the conditional statement in this case is true.

When *p* is false, I did not eat too much. When *q* is false, I will not get a stomach ache. It could be true that when I did not eat too much, I will not get a stomach ache. So, the truth value of the conditional statement in this case is true.

16.2 Writing the Converse of a Conditional Statement

The converse of a conditional statement of the form "If p, then q" is the statement of the form "If q, then p." The converse is a new statement that results when the hypothesis and conclusion of the conditional statement are switched. If a conditional statement is true, the converse may or may not be true.

Example

The converse of the conditional statement is shown.

Conditional Statement: "If a person is tall, then that person is a good basketball player."

Converse: If a person is a good basketball player, then that person is tall.

Both the conditional statement and the converse are false.

16.2 Writing the Inverse of a Conditional Statement

The inverse of a conditional statement of the form "If p, then q" is the statement of the form "If not p, then not q." The inverse is a new statement that results when the hypothesis and conclusion of the conditional statement are negated.

Example

The inverse of the conditional statement is shown.

Conditional Statement: "If Karen drives to school, then she will be on time."

Inverse: If Karen does not drive to school, then she will not be on time.

16.2 Writing the Contrapositive of a Conditional Statement

The contrapositive of a conditional statement of the form "If p, then q" is the statement of the form "If not q, then not p." The contrapositive is a new statement that results when the hypothesis and conclusion of the conditional statement are negated and switched.

Example

The contrapositive of the conditional statement is shown.

Conditional Statement: "If the sun is up, then it is daytime."

Contrapositive: If it is not daytime, then the sun is not up.

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16.2 Writing a Biconditional Statement

When a conditional statement and its converse are both true, they can be combined and written as a single statement using "if and only if." This new statement is called a biconditional statement.

Example

The biconditional statement of the conditional statement is shown.

Conditional Statement: "If a figure is an equilateral triangle, then that figure has three sides of equal length."

Converse: If a figure has three sides of equal length, then that figure is an equilateral triangle.
Biconditional Statement: A figure is an equilateral triangle if and only if that figure has three sides of equal length.

16.3 Using a Direct Proof

Some conditional statements can be proven using a direct proof by listing the logical steps and reasoning to get from the hypothesis to the conclusion.

Example

Conditional statement: If $a(b + c) + b(a + c) = c(b + a)$, then $a = 0$ or $b = 0$.

Steps	Reasons
$ab + ac + ba + bc = cb + ca$	Distributive Property
$ab + ac + ab + bc = bc + ac$	Commutative Property of Multiplication
$2ab + ac + bc = bc + ac$	Combine like terms.
$2ab + ac + bc = ac + bc$	Commutative Property of Addition
$2ab + ac + bc - bc = ac + bc - bc$	Algebraic equations are still true if you perform the same operation on both sides of an equation.
$2ab + ac = ac$	Inverse Property of Addition
$2ab + ac - ac = ac - ac$	Algebraic equations are still true if you perform the same operation on both sides of an equation.
$2ab = 0$	Inverse Property of Addition
$a = 0$ or $b = 0$	If a product is equal to zero, at least one factor in the product is equal to zero.

16.3 Using an Indirect Proof

Some conditional statements can be proven using an indirect proof. An indirect proof is also called proof by contradiction. To prove a statement using proof by contradiction, assume that the conclusion is false and then show that the hypothesis is false or state a contradiction. This is equivalent to showing that if the hypothesis is true, then the conclusion is also true.

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Example

Conditional statement: If $a(b + c) + b(a + c) = c(b + a)$, then $a = 0$ or $b = 0$.

$1(2 + c) + 2(1 + c) = c(2 + 1)$	Assume $a = 1$ and $b = 2$. (negation of conclusion)
$2 + c + 2 + 2c = 2c + c$	Distributive Property
$4 + 3c = 3c$	Combine like terms.
$4 + 3c - 3c = 3c - 3c$	Algebraic equations are still true if you perform the same operation on both sides of an equation.
$4 = 0$	Inverse Property of Addition

This is a contradiction because $4 \neq 0$. So, the conditional statement must be true.

16.4 Solving Problems Using Logic

Many problems are solved using logical thinking. Use the clues in the problem to list and eliminate possible solutions. Often problems will need to be worked out backwards or out of order.

Example

Sharon was thinking of a number. To help Rico guess the number, she gave him these clues. The number is less than 20 and a multiple of 6. He still couldn't be sure of the number, so she told him that the sum of the digits of the correct number is the square of the sum of the digits of another possible number.

The numbers that are less than 20 and a multiple of 6 are 6, 12, and 18. Because the final clue talks about the sum of the digits, the number must have more than 1 digit, so 6 is not the number. The sum of the digits in 12 is 3, and the sum of the digits in 18 is 9. The number 9 is the square of 3, so 18 is Sharon's number.

16.5 Solving Logic Puzzles Using Grids

To solve a logic puzzle, a grid is often used to keep the information organized. First, list each clue. Next, list the different categories and members of each category. If there are three categories with three possible members, use a 3 by 3 logic puzzle grid.

Example

Victor and his friends Ty and Nathan are starting up a band. Each friend will play the drums, the keyboard, or the guitar. Each friend will also write the lyrics, write the music, or sing. Victor is the lead singer. The friend who writes the music also plays the keyboard. Nathan plays the drums.

		Jobs			Instruments		
		Writes lyrics	Writes music	Sings	Keyboard	Guitar	Drums
Friends	Victor	X	X	O	X	O	X
	Ty	X	O	X	O	X	X
	Nathan	O	X	X	X	X	O
Instruments	Keyboard	X	O	X			
	Guitar	X	X	O			
	Drums	O	X	X			

Victor sings and plays the guitar.

Ty writes music and plays the keyboard.

Nathan writes lyrics and plays the drums.